Computational Learning Theory: Probably Approximately Correct (PAC) Learning
This lecture: Computational Learning Theory

• The Theory of Generalization

• Probably Approximately Correct (PAC) learning

• Positive and negative learnability results

• Agnostic Learning

• Shattering and the VC dimension
Where are we?

• The Theory of Generalization
  – When can be trust the learning algorithm?
  – What functions can be learned?
  – Batch Learning

• Probably Approximately Correct (PAC) learning

• Positive and negative learnability results

• Agnostic Learning

• Shattering and the VC dimension
This section

1. Define the PAC model of learning

2. Make formal connections to the principle of Occam’s razor
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2. Make formal connections to the principle of Occam’s razor
Formulating the theory of prediction

In the general case, we have

- \( X \): instance space, \( Y \): output space = \{+1, -1\}
- \( D \): an unknown distribution over \( X \)
- \( f \): an unknown target function \( X \rightarrow Y \), taken from a concept class \( C \)
- \( h \): a hypothesis function \( X \rightarrow Y \) that the learning algorithm selects from a hypothesis class \( H \)
- \( S \): a set of \( m \) training examples drawn from \( D \), labeled with \( f \)
- \( err_D(h) \): The true error of any hypothesis \( h \)
- \( err_S(h) \): The empirical error or training error or observed error of \( h \)
Theoretical questions

• Can we describe or bound the true error ($err_D$) given the empirical error ($err_S$)?

• Is a concept class $C$ learnable?

• Is it possible to learn $C$ using only the functions in $H$ using the supervised protocol?

• How many examples does an algorithm need to guarantee good performance?
Requirements of Learning

• Cannot expect a learner to learn a concept exactly
  – There will generally be multiple concepts consistent with the available data (which represent a small fraction of the available instance space)
  – Unseen examples could potentially have any label
  – We “agree” to misclassify uncommon examples that do not show up in the training set
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  – given small parameters $\epsilon$ and $\delta$,
  – With probability at least $1 - \delta$, a learner produces a hypothesis with error at most $\epsilon$. 
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- The only reason we can hope for this is the *consistent distribution assumption*.
Consider a concept class C defined over an instance space X (containing instances of length n), and a learner L using a hypothesis space H
PAC Learnability

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- For all $f \in C$,
- For all distribution $D$ over $X$, and fixed $0 < \varepsilon, \delta < 1$,
- Given $m$ examples sampled independently according to $D$, the algorithm $L$ produces, with probability at least $(1 - \delta)$, a hypothesis $h \in H$ that has error at most $\varepsilon$,

where $m$ is **polynomial** in $1/\varepsilon$, $1/\delta$, $n$ and $\text{size}(H)$. 

**Note:**

- $\Pr_D[f(x) \neq h(x)]$ denotes the probability that $f(x)$ and $h(x)$ disagree.

- In PAC learning, the goal is to find a hypothesis that generalizes well to the entire distribution $D$. 
- The parameters $m$, $\varepsilon$, and $\delta$ are used to control the trade-off between the number of examples needed and the confidence in the result.
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recall that $\text{Err}_D(h) = \Pr_D[f(x) \neq h(x)]$
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The concept class $C$ is **efficiently learnable** if $L$ can produce the hypothesis in time that is polynomial in $1/\varepsilon$, $1/\delta$, $n$ and size($H$).
PAC Learnability

- We impose two limitations

  - Polynomial sample complexity (information-theoretic constraint)
    - Is there enough information in the sample to distinguish a hypothesis $h$?

  - Polynomial time complexity (computational complexity)
    - Is there an efficient algorithm that can process the sample and produce a good hypothesis $h$?

To be PAC learnable, there must be a hypothesis $h \in H$ with arbitrary small error for every $f \in C$. We assume $H = C$. (Properly PAC learnable if $H = C$)

Worst case definition: the algorithm must meet its accuracy for every distribution (The distribution-free assumption) for every target function $f$ in the class $C$.
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  – for every target function \( f \) in the class \( C \)