Computational Learning Theory: The Theory of Generalization

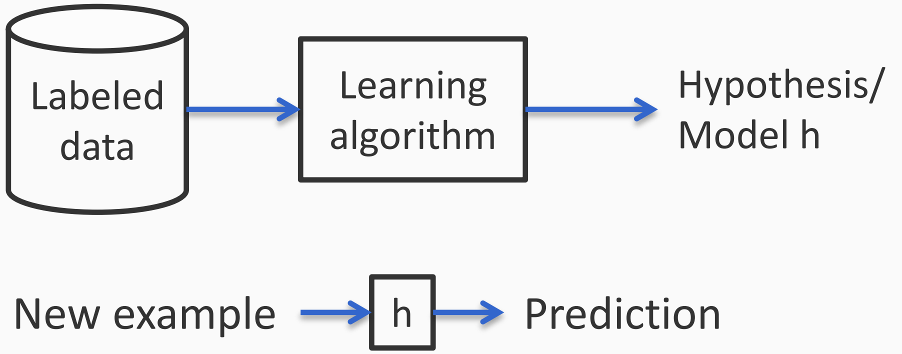
Machine Learning

THE UNIVERSITY OF UTAH

Slides based on material from Dan Roth, Avrim Blum, Tom Mitchell and others

Checkpoint: The bigger picture

- Supervised learning: instances, concepts, and hypotheses



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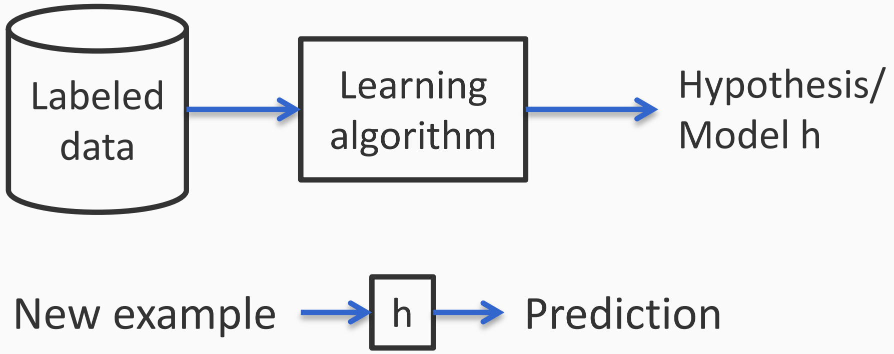
- Specific learners
  - Decision trees
  - Perceptron



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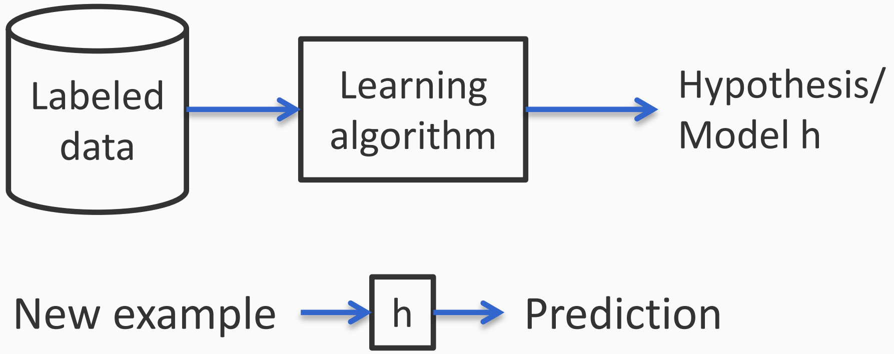

- General ML ideas
  - Features as high dimensional vectors
  - Overfitting
  - Mistake-bound: One way of asking "Can my problem be learned?"

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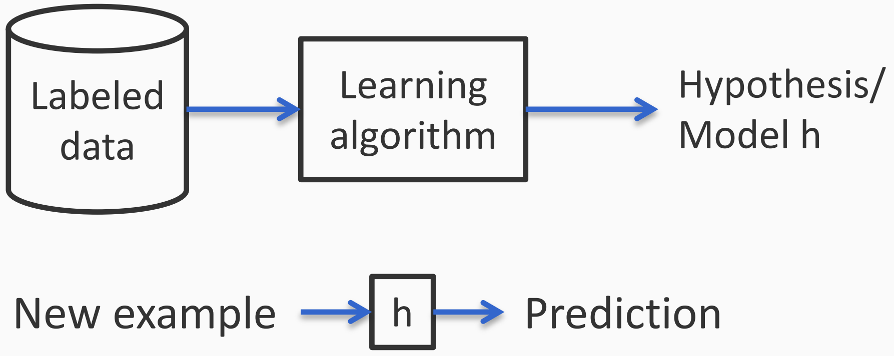

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Questions?

Computational Learning Theory

- The Theory of Generalization
- Probably Approximately Correct (PAC) learning
- Positive and negative learnability results
- Agnostic Learning
- Shattering and the VC dimension

This lecture: Computational Learning Theory

- The Theory of Generalization
  - When can be trust the learning algorithm?
  - Errors of hypotheses
  - Batch Learning
- Probably Approximately Correct (PAC) learning
- Positive and negative learnability results
- Agnostic Learning
- Shattering and the VC dimension

Computational Learning Theory

Are there general "laws of nature" related to learnability?

We want theory that can relate

- Probability of successful Learning
- Number of training examples
- Complexity of hypothesis space
- Accuracy to which target concept is approximated
- Manner in which training examples are presented

$$f = x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_{100}$$

Learning Conjunctions

Some random source (nature) provides training examples

Teacher (Nature) provides the labels (f(x))

- <(1,1,1,1,1,1,…,1,1), 1>
- <(1,1,1,0,0,0,…,0,0), 0>
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Notation: <example, label>

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For a reasonable learning algorithm (by *elimination*), the final hypothesis will be

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With the given data, we only learned an *approximation* to the true concept. Is it good enough?

Two Directions for How good is our learning algorithm?

- Can analyze the probabilistic intuition
  - Never saw $x_1=0$ in positive examples, maybe we'll never see it
  - And if we do, it will be with small probability, so the concepts we learn may be *pretty good*
    - *Pretty good:* In terms of performance on future data
  - ***PAC framework***

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- ***Mistake Driven*** Learning algorithms
  - Update your hypothesis only when you make mistakes
  - Define *good* in terms of how many mistakes you make before you stop

The mistake bound approach

The mistake bound model is a theoretical approach

- We may be able to determine the number of mistakes the learning algorithm can make before converging

But no answer to *"How many examples do you need before converging to a good hypothesis?"*

Because the mistake-bound model makes no assumptions about the order or distribution of training examples

- Both a strength and a weakness of the mistake bound model

PAC learning

A model for *batch learning*

- Train on a fixed training set
- Then deploy it in the wild

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How well will your learning algorithm do on future instances after it was trained on the fixed training set?

How big should the training set be to learn some concept?

Can we guarantee that learning will succeed?

The setup

- Instance Space: X, the set of examples

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- Training instances: $S \times \{-1,1\}$: positive and negative examples of the target concept. (S is a finite subset of X)

$$\left(x_1, f(x_1)\right), \left(x_2, f(x_2)\right), \cdots, \left(x_n, f(x_n)\right)$$

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…for all $x \in S$? *Or* …for all $x \in X$?

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  - This is the set that the learning algorithm explores
- Training instances: $S \times \{-1,1\}$: positive and negative examples of the target concept. (S is a finite subset of X)
  - *Training instances are generated by a fixed unknown probability distribution D over X*
- What we want: A hypothesis $\mathrm{h} \in H$ such that $h(x) = f(x)$
  - Evaluate h on subsequent examples $x \in X$ drawn according to D

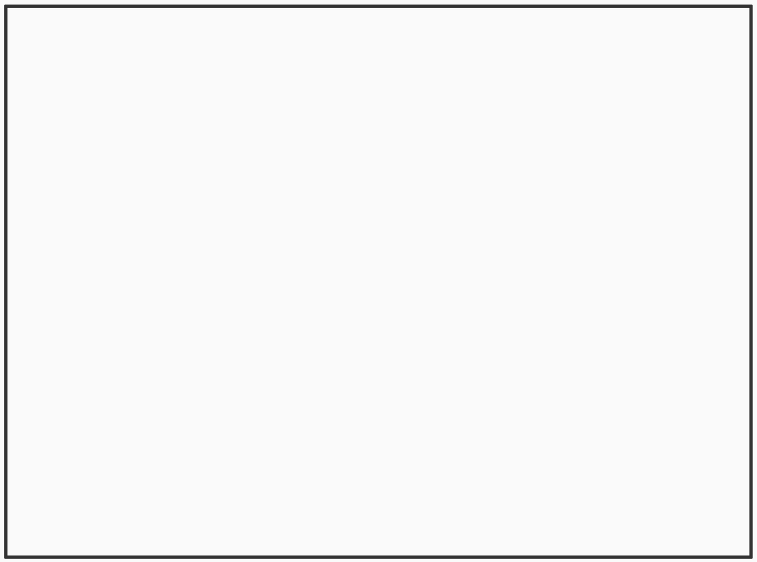
Distribution over the instance space

Consider a two dimensional instance space

Not all points in the space are equally likely to exist as instances.

> For example, not every sequence of words is an email, not every sequence of letters is a name

Distribution over the instance space



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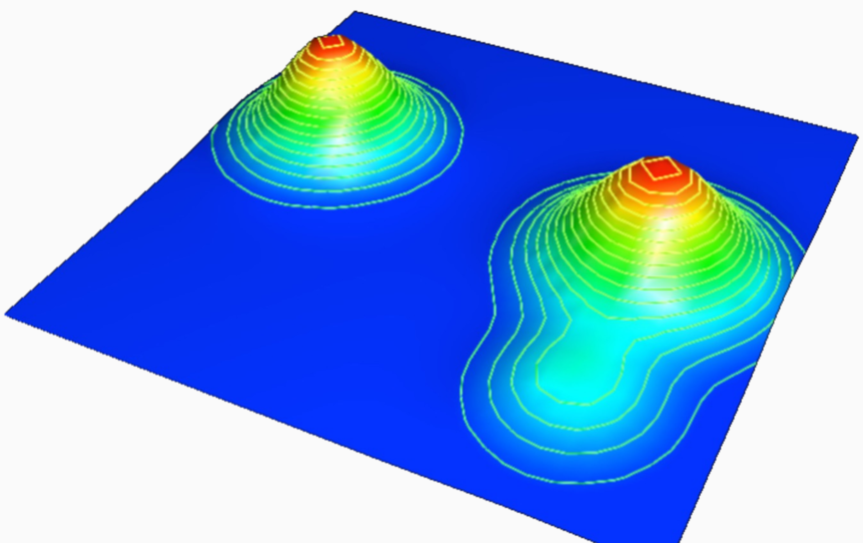

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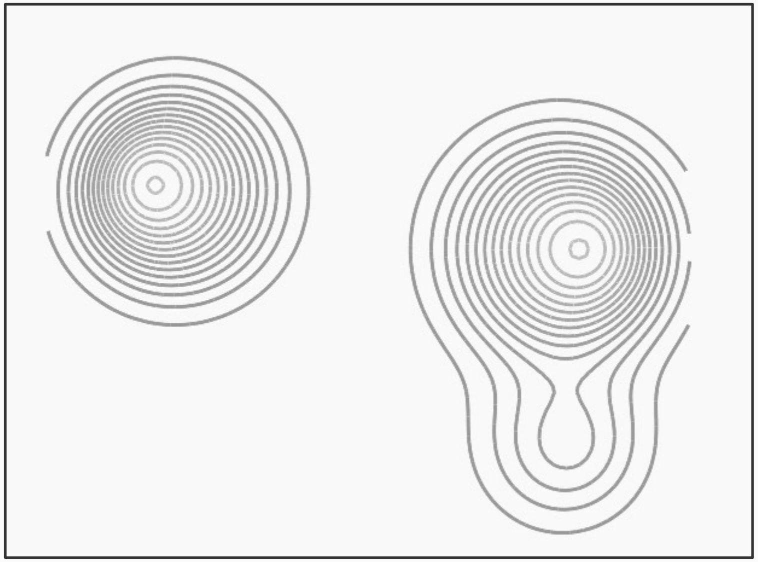
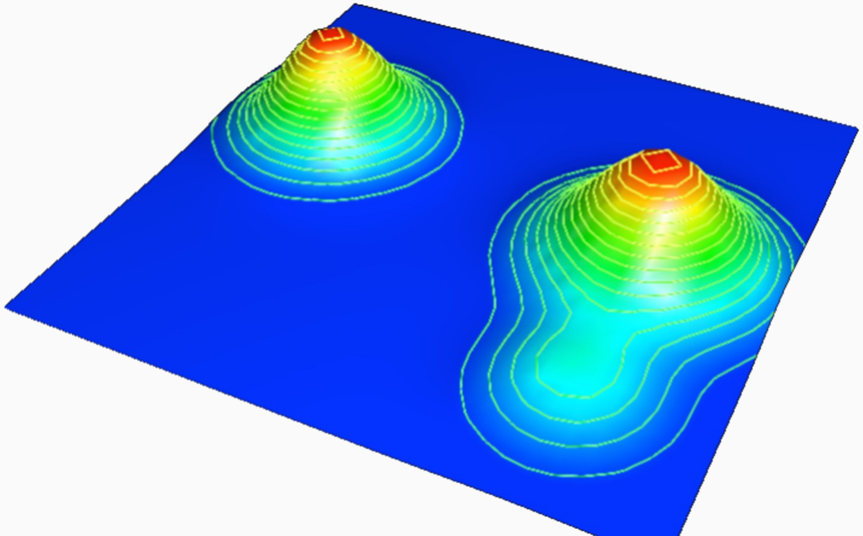
Or as a contour plot





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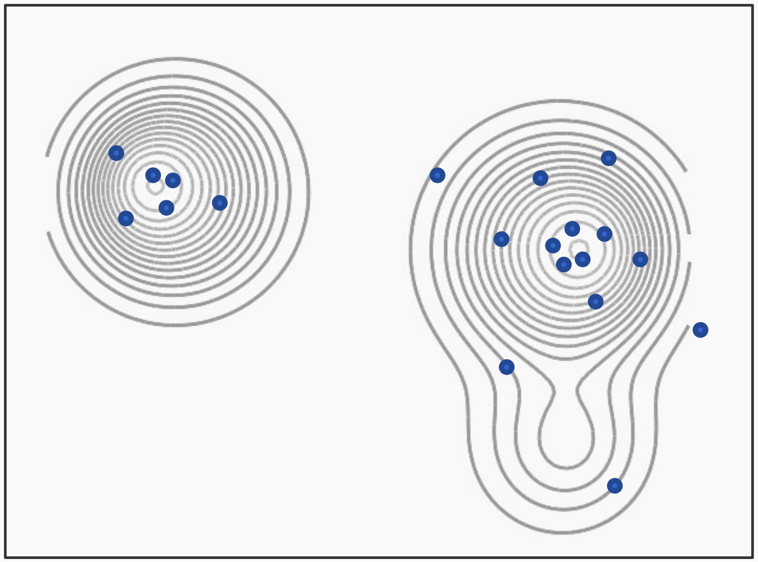

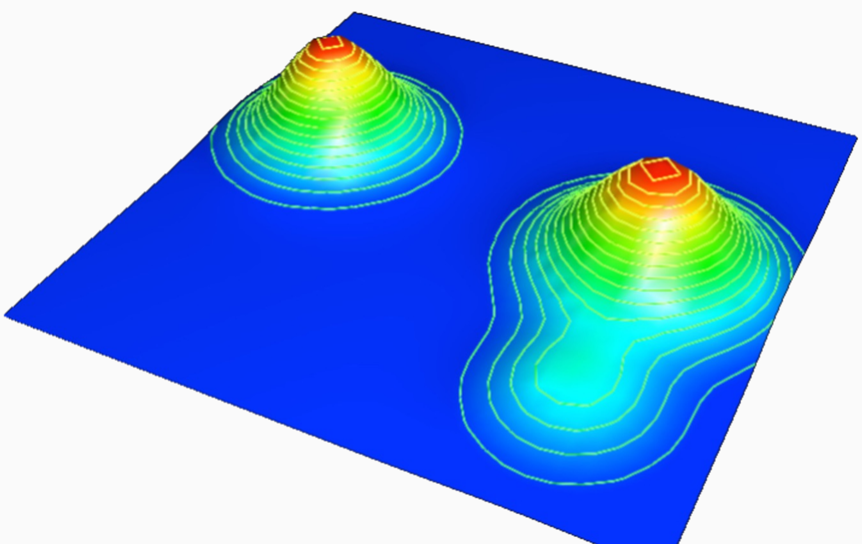

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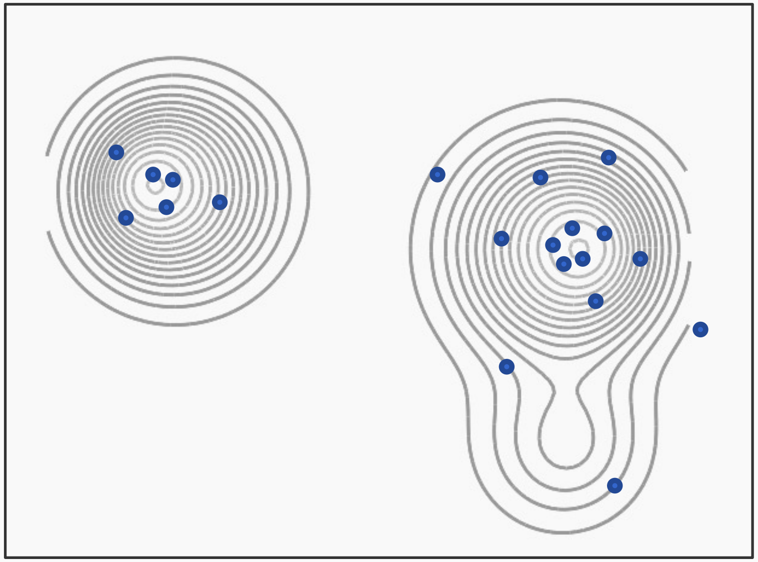

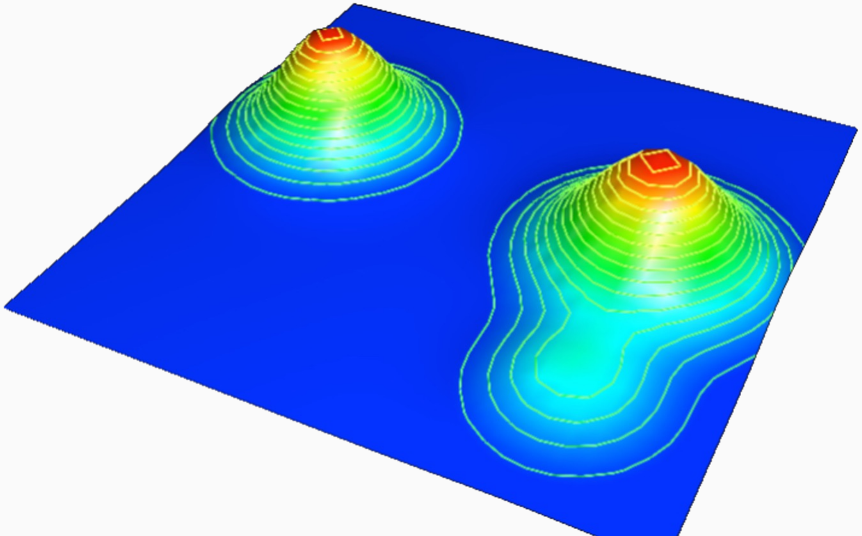

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*We may not know what the distribution is, but we assume one exists and **is fixed***

PAC Learning – Intuition

The assumption of fixed distribution is important for two reasons:

1. Gives us hope that what we learn on the training data will be meaningful on future examples
2. Also gives a well-defined notion of the error of a hypothesis according to the target function

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1. Gives us hope that what we learn on the training data will be meaningful on future examples
2. Also gives a well-defined notion of the error of a hypothesis according to the target function

"*The future will be like the past*": We have seen many examples (drawn according to the distribution D)

– Since in all the positive examples x_1 was active, it is very likely that it will be active in future positive examples
– If not, in any case, x_1 is active only in a small percentage of the examples so our error will be small

Error of a hypothesis

Definition

Given a distribution D over examples, the *error* of a hypothesis h with respect to a target concept f is

$$\mathrm{err}_D(h) = \Pr_{x\sim D}[h(x) \neq f(x)]$$

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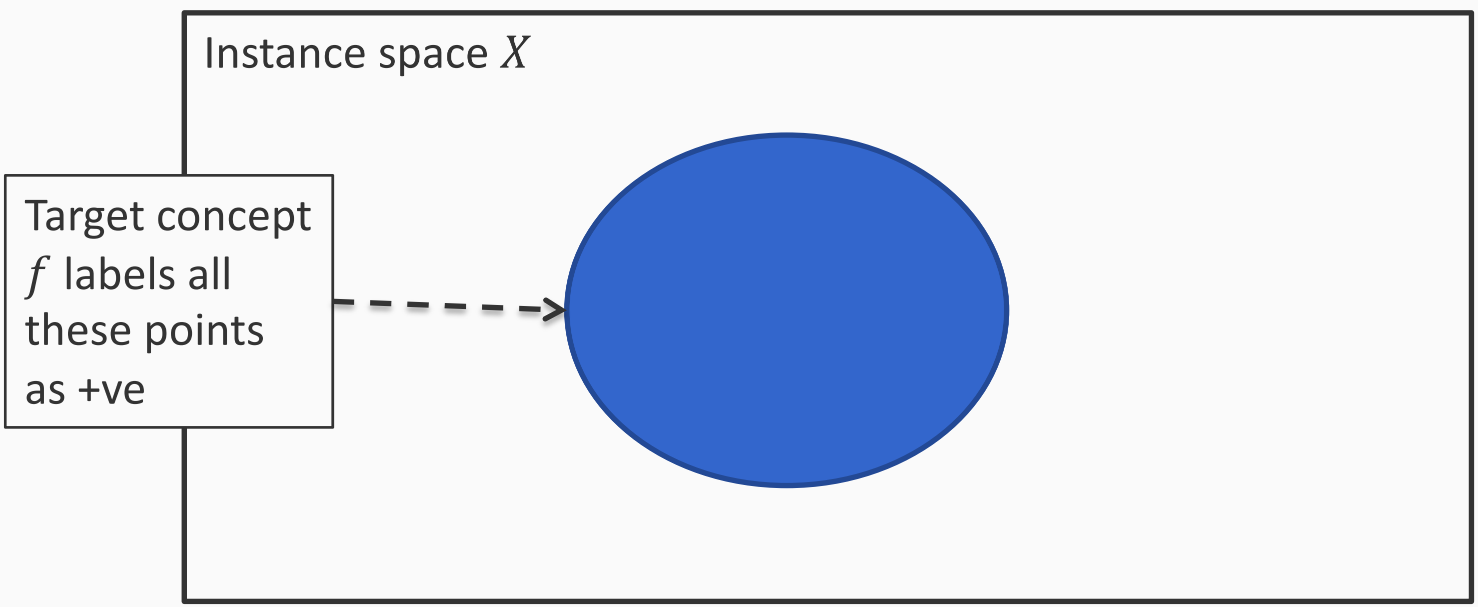

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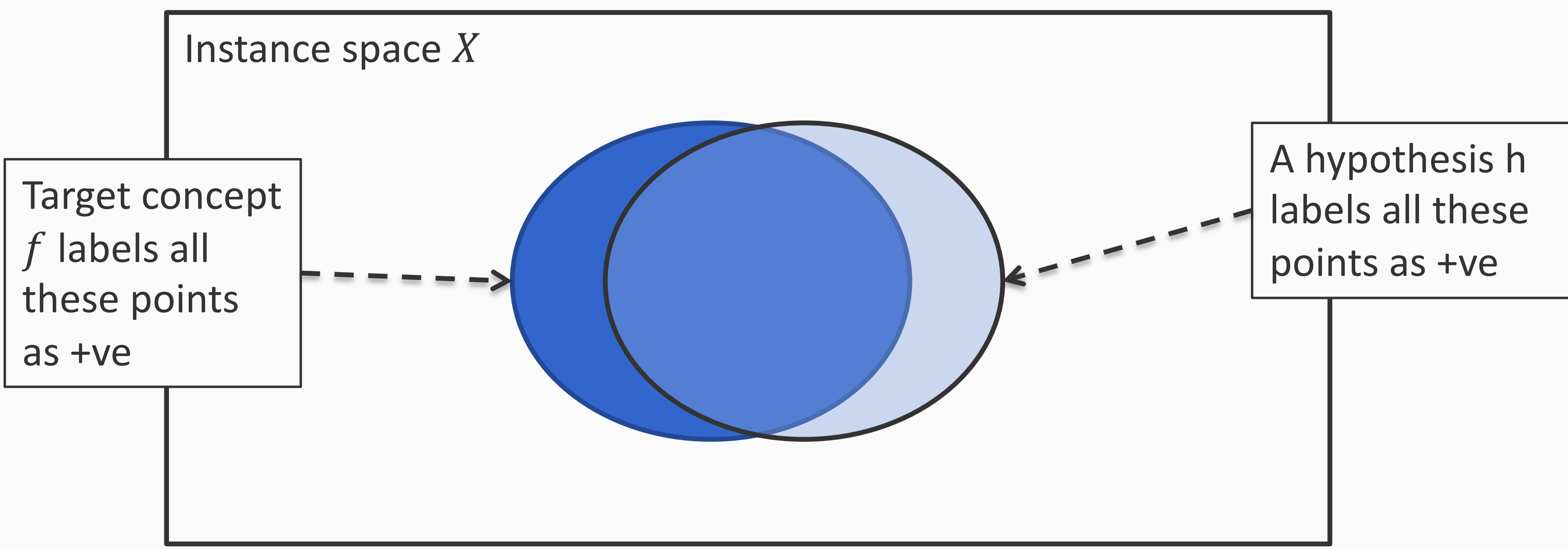

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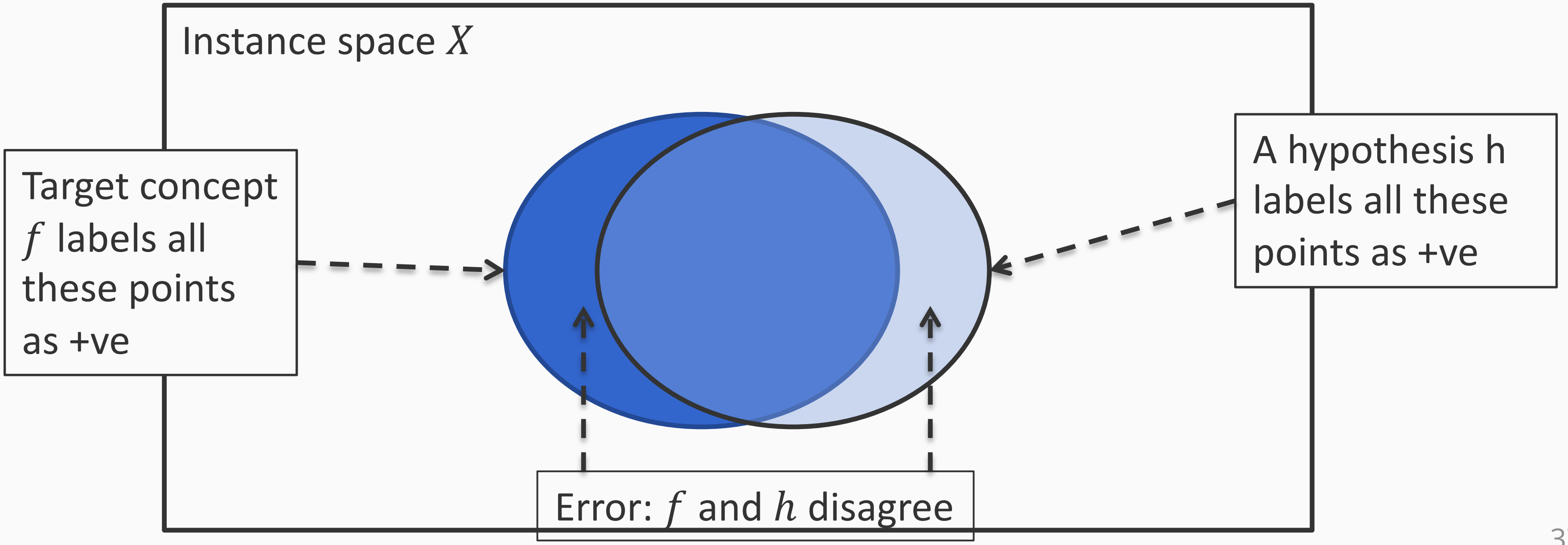

Empirical error

Contrast true error against the *empirical error*

For a target concept f, the empirical error of a hypothesis h is defined for a training set S as the fraction of examples x in S for which the functions f and h disagree. That is, $h(x) \neq f(x)$

Denoted by $\mathrm{err}_S(h)$

Overfitting: When the empirical error on the training set $\mathrm{err}_S(h)$ is substantially lower than $\mathrm{err}_D(h)$

The goal of batch learning

To devise good learning algorithms that avoid overfitting

Not fooled by functions that only appear to be good because they explain the training set very well

Online learning vs. Batch learning

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- No assumptions about the distribution of examples

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- Learning is a sequence of trials
  - Learner sees a single example, makes a prediction
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- Goal: To bound the total number of mistakes over time (for mistake-bound learning)

Batch learning

- Examples are drawn from a fixed (perhaps unknown) probability distribution D over the instance space
- Learning uses a training set S, drawn i.i.d from the distribution D
- Goal: To find a hypothesis that has low chance of making a mistake on a new example from D