Computational Learning Theory: Positive and negative learnability results

Machine Learning

THE UNIVERSITY OF UTAH

Slides based on material from Dan Roth, Avrim Blum, Tom Mitchell and others

Computational Learning Theory

- The Theory of Generalization
- Probably Approximately Correct (PAC) learning
- Positive and negative learnability results
- Agnostic Learning
- Shattering and the VC dimension

This lecture: Computational Learning Theory

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What can be learned

General conjunctions are PAC learnable

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- Increasing the confidence to 99% will cost 1145 examples (logarithmic with δ)

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3-CNF

$$(l_{11} \vee l_{12} \vee l_{13}) \wedge (l_{21} \vee l_{22} \vee l_{23})$$

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For PAC learnability, we still need an efficient algorithm that will find a consistent hypothesis.
Exercise: Find one

$$m > \frac{1}{\epsilon}\left(\ln(|H|) + \ln\frac{1}{\delta}\right)$$

What can be learned

General Boolean functions

- How many Boolean functions exist with n variables?

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- How many Boolean functions exist with n variables? 2^{2^n}

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  So $\log(|H|)$ is exponential.
- General Boolean functions are not PAC learnable

Sample Complexity

- k-CNF: Conjunctions of any number of clauses where each disjunctive clause has at most k literals.
- k-clause-CNF: Conjunctions of at most k disjunctive clauses.

$$f = C_1 \wedge C_2 \wedge \cdots \wedge C_k$$
$$C_i = l_1 \vee l_2 \vee \cdots \vee l_m$$

$$\ln(|\text{k−clause−CNF}|) = O(kn)$$

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All these classes can be learned using a polynomial size sample

Exercise: Prove that the above four classes of functions have polynomial sample complexity

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Suppose we want to learn a 2-term-DNF
What should our hypothesis class be?

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- But, we have seen an algorithm for learning k-CNF (It was an exercise a few slides back.)

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Example:

$(a \wedge b \wedge c) \vee (d \wedge e \wedge f) = (a \vee d) \wedge (a \vee e) \wedge (a \vee f) \wedge (b \vee d) \wedge \cdots \wedge (c \vee f)$

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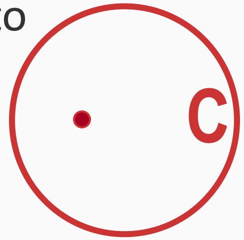

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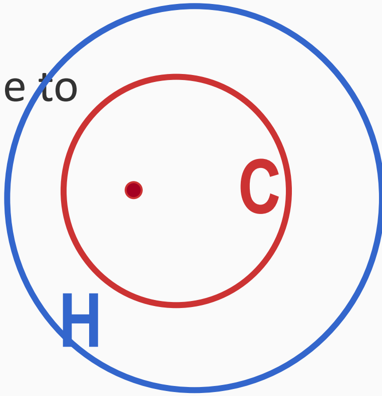

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The lesson: Importance of representation Concepts that cannot be learned using one representation can *sometimes* be learned using a different, more expressive, representation

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We have seen this idea before: Linear classifiers for conjunctions

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   - Showing that various concepts classes cannot be learned, based on well-accepted assumptions from computational complexity theory
   - Takes the form “A concept class C cannot be learned unless P=NP”

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Two types of non-learnability results

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   - Showing that various concepts classes cannot be learned, based on well-accepted assumptions from computational complexity theory
   - Takes the form "A concept class C cannot be learned unless P=NP"

2. Information Theoretic (sample complexity bad)
   - The concept class is sufficiently rich that a polynomial number of examples may not be sufficient to distinguish a particular target concept
   - The proof typically shows that a given class cannot be learned by algorithms using hypotheses from the same class. (Is this always a problem?)

Negative Results for Learning

- Complexity Theoretic
  - k-term DNF, for k>1 (k-clause CNF, k>1)
  - Neural Networks of fixed architecture: 2-layer, 3-nodes, n inputs, threshold activations. (Blum and Rivest, 1988)
  - “read-once” Boolean formulas (Pitt and Valiant, 1988)
  - Quantified conjunctive concepts

- Information Theoretic
  - Arbitrary Boolean functions (DNF Formulas or CNF Formulas)
  - Deterministic Finite Automata
  - Context Free Grammars