Multiplicative Updates
&
the Winnow Algorithm

Machine Learning

THE UNIVERSITY OF UTAH

Where are we?

- Still looking at linear classifiers
- Still looking at mistake-bound learning

- We have seen the Perceptron update rule

> - Receive an input $(\boldsymbol{x}_i, y_i)$
> - if $\text{sgn}(\mathbf{w}_t^T \boldsymbol{x}_i) \neq y_i$:
>   Update $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + y_i \, \boldsymbol{x}_i$

- The Perceptron update is an example of an *additive weight update*

This lecture

- The Winnow Algorithm
- Winnow mistake bound
- Generalizations

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The setting

- Recall linear threshold units
  - Prediction = +1 if $w^T \boldsymbol{x} \geq \theta$
  - Prediction = -1 if $w^T \boldsymbol{x} < \theta$

- The Perceptron mistake bound is $(R/\gamma)^2$
  - For Boolean functions with n attributes, $R^2 = n$, so basically O(n)

- *Motivating question*:

  Suppose we know that even though the number of attributes is n, the number of relevant attributes is k, which is much less than n

  Can we improve the mistake bound?

Learning when irrelevant attributes abound

Example

- Suppose we know that the true concept is a disjunction of only a small number of features
  - Say only x_1 and x_2 are relevant

Learning when irrelevant attributes abound

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- Suppose we know that the true concept is a disjunction of only a small number of features
  - Say only x_1 and x_2 are relevant
- The elimination algorithm will work:
  - Start with h(x) = $x_1 \vee x_2 \vee \cdots \vee x_{1024}$
  - Mistake on a negative example: Eliminate all attributes in the example from your hypothesis function h
    - Suppose we have an example x_{100} = 1, x_{301} = 1, label = -1
    - Simple update: just eliminate these two variables from the function
  - Will never make mistakes on a positive example. Why?

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- Makes O(n) updates

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- But we know that our function is a k-disjunction (here k = 2)
  - And there are only C(n, k) · 2^k $\approx$ $n^k 2^k$ such functions
  - The Halving algorithm will make k log(n) mistakes
  - Can we realize this bound with an efficient algorithm?

Multiplicative updates

- Let’s use linear classifiers with a different update rule
  - Remember: Perceptron will make O(n) mistakes on Boolean functions

- The idea: Weights should be promoted and demoted via multiplicative, rather than additive, updates

The Winnow algorithm

Littlestone 1988

Given a training set D = $\{(\mathbf{x}, y)\}$, $\mathbf{x} \in \{0,1\}^n$, $y \in \{-1,1\}$

The Winnow algorithm

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1. Initialize: **w** = $(1,1,1,1\ldots,1) \in \Re^{n}$, θ = n

The Winnow algorithm

Littlestone 1988

Given a training set D = {($\mathbf{x}$, y)}, $\mathbf{x} \in$ {0,1}^{n}, $y \in$ {-1,1}

1. Initialize: **w** = (1,1,1,1…,1) $\in \Re^{n}$, θ = n

2. For each training example ($\mathbf{x}$, y):
   - Predict y' = sgn($\mathbf{w}^T\mathbf{x} - \theta$)

The Winnow algorithm

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Given a training set D = $\{(\mathbf{x}, y)\}$, $\mathbf{x} \in \{0,1\}^n$, $y \in \{-1,1\}$

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2. For each training example $(\mathbf{x}, y)$:
   - Predict $y' = \text{sgn}(\mathbf{w}^T\mathbf{x} - \theta)$
   - If $y = +1$ and $y' = -1$ then:

Promotion
     - Update each weight $w_i \leftarrow 2w_i$ *only* for those features $\mathbf{x}_i$ that are 1

The Winnow algorithm

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     - Update each weight $w_i \leftarrow 2w_i$ *only* for those features $\mathbf{x}_i$ that are 1

     Else if $y = -1$ and $y' = +1$ then:

     Demotion
     - Update each weight $w_i \leftarrow w_i/2$ *only* for those features $\mathbf{x}_i$ that are 1

Example run of the algorithm

$f = x_1 \vee x_2 \vee x_{1023} \vee x_{1024}$ Initialize $\theta = 1024$, $\mathbf{w} = (1,1,1,1\ldots,1)$

Example run of the algorithm

f = $x_1 \vee x_2 \vee x_{1023} \vee x_{1024}$ Initialize θ = 1024, **w** = (1,1,1,1…,1)

Example	**Prediction**	**Error?**	**Weights**
$\boldsymbol{x}$=(1,1,1,…,1), y=+1	$\mathbf{w}^{\mathsf{T}}\boldsymbol{x} \geq \theta$	No	**w** = (1,1,1,1…,1)

Example run of the algorithm

$f = x_1 \vee x_2 \vee x_{1023} \vee x_{1024}$ Initialize θ = 1024, **w** = (1,1,1,1…,1)

Example	Prediction	Error?	Weights
x=(1,1,1,…,1), y=+1	$\mathbf{w}^{\mathsf{T}}x \geq \theta$	No	**w** = (1,1,1,1…,1)
x=(0,0,0,…,0), y=−1	$\mathbf{w}^{\mathsf{T}}x < \theta$	No	**w** = (1,1,1,1…,1)

Example run of the algorithm

$f = x_1 \vee x_2 \vee x_{1023} \vee x_{1024}$ Initialize θ = 1024, **w** = (1,1,1,1…,1)

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x=(0,0,1,1,1,…,0), y=−1	$\mathbf{w}^T x < \theta$	No	**w** = (1,1,1,1…,1)

No changes until there are mistakes

Example run of the algorithm

$f = x_1 \vee x_2 \vee x_{1023} \vee x_{1024}$ Initialize θ = 1024, **w** = (1,1,1,1…,1)

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x=(1,1,1,…,1), y=+1	$\mathbf{w}^{\mathsf{T}}x \geq \theta$	No	**w** = (1,1,1,1…,1)
x=(0,0,0,…,0), y=−1	$\mathbf{w}^{\mathsf{T}}x < \theta$	No	**w** = (1,1,1,1…,1)
x=(0,0,1,1,1,…,0), y=−1	$\mathbf{w}^{\mathsf{T}}x < \theta$	No	**w** = (1,1,1,1…,1)
x=(1,0,0,…,0), y=+1	$\mathbf{w}^{\mathsf{T}}x < \theta$	Yes	**w** = (**2**,1,1,1…,1)

Promote x_1

Example run of the algorithm

$f = x_1 \vee x_2 \vee x_{1023} \vee x_{1024}$ Initialize $\theta = 1024$, $\mathbf{w} = (1,1,1,1\ldots,1)$

Example	Prediction	Error?	Weights
x=(1,1,1,…,1), y=+1	$\mathbf{w}^{\mathsf{T}}x \geq \theta$	No	$\mathbf{w}$ = (1,1,1,1…,1)
x=(0,0,0,…,0), y=−1	$\mathbf{w}^{\mathsf{T}}x < \theta$	No	$\mathbf{w}$ = (1,1,1,1…,1)
x=(0,0,1,1,1,…,0), y=−1	$\mathbf{w}^{\mathsf{T}}x < \theta$	No	$\mathbf{w}$ = (1,1,1,1…,1)
x=(1,0,0,…,0), y=+1	$\mathbf{w}^{\mathsf{T}}x < \theta$	Yes	$\mathbf{w}$ = (**2**,1,1,1…,1)
x=(0,1,0,…,0), y=+1	$\mathbf{w}^{\mathsf{T}}x < \theta$	Yes	$\mathbf{w}$ = (2,**2**,1,1…,1)

Promote x_2

Example run of the algorithm

f = $x_1 \vee x_2 \vee x_{1023} \vee x_{1024}$ Initialize θ = 1024, **w** = (1,1,1,1…,1)

Example	Prediction	Error?	Weights
x=(1,1,1,…,1), y=+1	$\mathbf{w}^T x \geq \theta$	No	**w** = (1,1,1,1…,1)
x=(0,0,0,…,0), y=−1	$\mathbf{w}^T x < \theta$	No	**w** = (1,1,1,1…,1)
x=(0,0,1,1,1,…,0), y=−1	$\mathbf{w}^T x < \theta$	No	**w** = (1,1,1,1…,1)
x=(1,0,0,…,0), y=+1	$\mathbf{w}^T x < \theta$	Yes	**w** = (**2**,1,1,1…,1)
x=(0,1,0,…,0), y=+1	$\mathbf{w}^T x < \theta$	Yes	**w** = (2,**2**,1,1…,1)
x=(1,1,1,…,0), y=+1	$\mathbf{w}^T x < \theta$	Yes	**w** = (**4**,**4**,**2**,1…,1)

Promote x_1, x_2 and x_3

Example run of the algorithm

$f = x_1 \lor x_2 \lor x_{1023} \lor x_{1024}$ Initialize θ = 1024, **w** = (1,1,1,1…,1)

Example	Prediction	Error?	Weights
x=(1,1,1,…,1), y=+1	$\mathbf{w}^T x \geq \theta$	No	**w** = (1,1,1,1…,1)
x=(0,0,0,…,0), y=−1	$\mathbf{w}^T x < \theta$	No	**w** = (1,1,1,1…,1)
x=(0,0,1,1,1,…,0), y=−1	$\mathbf{w}^T x < \theta$	No	**w** = (1,1,1,1…,1)
x=(1,0,0,…,0), y=+1	$\mathbf{w}^T x < \theta$	Yes	**w** = (**2**,1,1,1…,1)
x=(0,1,0,…,0), y=+1	$\mathbf{w}^T x < \theta$	Yes	**w** = (2,**2**,1,1…,1)
x=(1,1,1,…,0), y=+1	$\mathbf{w}^T x < \theta$	Yes	**w** = (**4**,**4**,**2**,1…,1)
x=(1,0,0,…,1), y=+1	$\mathbf{w}^T x < \theta$	Yes	**w** = (**8**,4,2,1…,**2**)
…	…	…	…

Example run of the algorithm

$f = x_1 \vee x_2 \vee x_{1023} \vee x_{1024}$ Initialize θ = 1024, **w** = (1,1,1,1…,1)

Example	**Prediction**	**Error?**	**Weights**
x=(1,1,1,…,1), y=+1	$\mathbf{w}^\top x \geq \theta$	No	**w** = (1,1,1,1…,1)
x=(0,0,0,…,0), y=−1	$\mathbf{w}^\top x < \theta$	No	**w** = (1,1,1,1…,1)
x=(0,0,1,1,1,…,0), y=−1	$\mathbf{w}^\top x < \theta$	No	**w** = (1,1,1,1…,1)
x=(1,0,0,…,0), y=+1	$\mathbf{w}^\top x < \theta$	Yes	**w** = (**2**,1,1,1…,1)
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x=(1,0,0,…,1), y=+1	$\mathbf{w}^\top x < \theta$	Yes	**w** = (**8**,4,2,1…,**2**)
…	…	…	…

Suppose after many steps, **w** = (**512**,**256**,512,512…,**512**)

Example run of the algorithm

$f = x_1 \vee x_2 \vee x_{1023} \vee x_{1024}$ Initialize θ = 1024, **w** = (1,1,1,1…,1)

Example	Prediction	Error?	Weights
x=(1,1,1,…,1), y=+1	$\mathbf{w}^{\mathsf{T}}x \geq \theta$	No	**w** = (1,1,1,1…,1)
x=(0,0,0,…,0), y=−1	$\mathbf{w}^{\mathsf{T}}x < \theta$	No	**w** = (1,1,1,1…,1)
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…	…	…	…
			w = (**512**,**256**,512,512…,**512**)
x=(0,0,1,1,…,0), y=−1	$\mathbf{w}^{\mathsf{T}}x \geq \theta$	Yes	**w** = (512,256,**256**,**256**…,512)

Demote x_3 and x_4

Example run of the algorithm

$f = x_1 \vee x_2 \vee x_{1023} \vee x_{1024}$ Initialize θ = 1024, **w** = (1,1,1,1…,1)

Example	Prediction	Error?	Weights
x=(1,1,1,…,1), y=+1	$\mathbf{w}^T x \geq \theta$	No	**w** = (1,1,1,1…,1)
x=(0,0,0,…,0), y=−1	$\mathbf{w}^T x < \theta$	No	**w** = (1,1,1,1…,1)
x=(0,0,1,1,1,…,0), y=−1	$\mathbf{w}^T x < \theta$	No	**w** = (1,1,1,1…,1)
x=(1,0,0,…,0), y=+1	$\mathbf{w}^T x < \theta$	Yes	**w** = (**2**,1,1,1…,1)
x=(0,1,0,…,0), y=+1	$\mathbf{w}^T x < \theta$	Yes	**w** = (2,**2**,1,1…,1)
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…	…	…	…
			w = (**512**,**256**,512,512…,**512**)
x=(0,0,1,1,…,0), y=−1	$\mathbf{w}^T x \geq \theta$	Yes	**w** = (512,256,**256**,**256**…,512)
x=(0,0,0,…,1), y=+1	$\mathbf{w}^T x < \theta$	Yes	**w** = (512,256,256,256…,**1024**)

Example run of the algorithm

$f = x_1 \vee x_2 \vee x_{1023} \vee x_{1024}$ Initialize θ = 1024, $\mathbf{w}$ = (1,1,1,1…,1)

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Eventually, the algorithm will converge to something like
$\mathbf{w}$ = (**1024**,**1024**,16,2…, **1024**,**1024**)

The multiplicative update

Widely used (and re-re-discovered) in various fields

- Winnow (and the Majority Weighted algorithm)
- We will see the AdaBoost algorithm
- Shows up in economics and game theory (from the 1950s)
- Computational Geometry
- Operations research
- Many more…

See: Sanjeev Arora, Elad Hazan and Satyen Kale, *The Multiplicative Weights Update Method: a Meta Algorithm and Applications,* for a survey

This lecture

- The Winnow Algorithm
- Winnow mistake bound
- Generalizations

The Winnow algorithm

Littlestone 1988

Given a training set D = $\{(\boldsymbol{x}, y)\}$, $\boldsymbol{x} \in \Re^n$, $y \in \{-1,1\}$

1. Initialize: $\mathbf{w} = (1,1,1,1\ldots,1) \in \Re^n$, $\theta = n$

2. For each training example $(\boldsymbol{x}, y)$:
   - Predict $y' = \mathrm{sgn}(\mathbf{w}^T\boldsymbol{x} - \theta)$
   - If $y = +1$ and $y' = -1$ then:
     - Update each weight $\mathbf{w}_i \leftarrow 2\mathbf{w}_i$ *only* for those features $\boldsymbol{x}_i$ that are 1

     Else if $y = -1$ and $y' = +1$ then:
     - Update each weight $\mathbf{w}_i \leftarrow \mathbf{w}_i/2$ *only* for those features $\boldsymbol{x}_i$ that are 1

Winnow mistake bound

We will analyze the simple case of *k-disjunctions*

Theorem

The Winnow algorithm learns the class of k-disjunctions with n Boolean variables in the Mistake bound model, making $O(k \log n)$ mistakes.

Winnow mistake bound

We will analyze the simple case of *k-disjunctions*

Theorem

The Winnow algorithm learns the class of k-disjunctions with n Boolean variables in the Mistake bound model, making O(k log n) mistakes.

Implications:

1. Recall: The Perceptron mistake bound is O(n), "throwing lots of features at the problem" can hurt learning

2. Winnow is *attribute efficient* because it only has a log dependency on n. Only a small penalty for trying out lots of features

Proof

Theorem: Winnow will make at most O(k log n) mistakes with k-disjunctions

Our target functions are k-disjunctions

Strategy

Total mistakes = mistakes on positive examples

+

mistakes on negative examples

Get mistake bound by upper bounding each separately

Proof

Theorem: Winnow will make at most O(k log n) mistakes with k-disjunctions

Our target functions are k-disjunctions

Strategy

Total mistakes = mistakes on positive examples m^+

+

mistakes on negative examples m^-

Get mistake bound by upper bounding each separately

1. Mistakes on positives

(The true label is positive, the prediction is negative)

Theorem: Winnow will make at most O(k log n) mistakes with k-disjunctions

Our target functions are k-disjunctions

- A mistake on a positive example will double the weights for *at least* one of the relevant attributes. Why?

  Because a positive example will have at least one relevant attribute

- We initialized our weight vector with 1's and the threshold θ is always fixed to n

- How many times can a relevant attribute get promoted (i.e. doubled) before it hits n?
  - 1 + log(n) times. After that, it will cross θ

Number of mistakes on positive examples $m^+ \leq k\,(1 + \log(n))$

2. Mistakes on negatives

(The true label is negative, the prediction is positive)

Theorem: Winnow will make at most O(k log n) mistakes with k-disjunctions

Our target functions are k-disjunctions

There is *no relevant feature* in the example, yet the dot product of weights and features was more than n

- Halve all the weights of the features in this example. *No relevant feature* will ever get demoted.

But will irrelevant features ever get promoted (i.e their weights doubled)?

- Yes. If an irrelevant feature shows up in a positive example, it may get promoted

Contrast: Relevant features will only get promoted and never demoted

- We need a different way to count mistakes on negatives

2. Mistakes on negatives

Let's use a different strategy

Theorem: Winnow will make at most O(k log n) mistakes with k-disjunctions

Our target functions are k-disjunctions

Track the sum of all weights over time: $TW_t = \sum_{i=1}^{n} w_i^t$

1. The weights are never negative, neither is their sum

$$TW_t > 0$$

2. The initial value of the sum is n (because all weights are initialized to 1)

$$TW_0 = n$$

2. Mistakes on negatives

Let's use a different strategy

Theorem: Winnow will make at most O(k log n) mistakes with k-disjunctions

Our target functions are k-disjunctions

Track the sum of all weights over time: $TW_t = \sum_{i=1}^{n} w_i^t$

3. What happens to TW_t when there is a mistake on a positive example?

$$TW_{t+1} < TW_t + n$$

Why?

Total increase because of positive examples $< \; nm^+$

2. Mistakes on negatives

Let's use a different strategy

Theorem: Winnow will make at most O(k log n) mistakes with k-disjunctions

Our target functions are k-disjunctions

Track the sum of all weights over time: $TW_t = \sum_{i=1}^{n} w_i^t$

4. What happens to TW_t when there is a mistake on a negative example?

$$TW_{t+1} < TW_t - \frac{n}{2}$$

Why?

Total *decrease* because of negative examples $< \frac{n}{2} m^{-}$

2. Mistakes on negatives

Let's use a different strategy

Theorem: Winnow will make at most O(k log n) mistakes with k-disjunctions

Our target functions are k-disjunctions

What we know:

$$TW_t > 0 \qquad TW_0 = n$$

Total increase because of positive examples $< nm^+$

Total decrease because of negative examples $< \frac{n}{2}m^-$

Putting these together: $0 < TW_t < n + nm^+ - \frac{n}{2}m^-$

Number of mistakes on negative examples $m^- < 2(1 + m^+)$

3. Mistake bound

Theorem: Winnow will make at most O(k log n) mistakes with k-disjunctions

Our target functions are k-disjunctions

Mistakes on positive examples: $m^+ \leq k\left(1 + \log(n)\right)$

Mistakes on negative examples: $m^- < 2(1 + m^+)$

Total number of mistakes = $m^+ + m^- < m^+ + 2(1 + m^+)$

$$< 2 + 3k\left(1 + \log(n)\right)$$

Number of mistakes Winnow will make on k-disjunctions = $O\left(k \log(n)\right)$

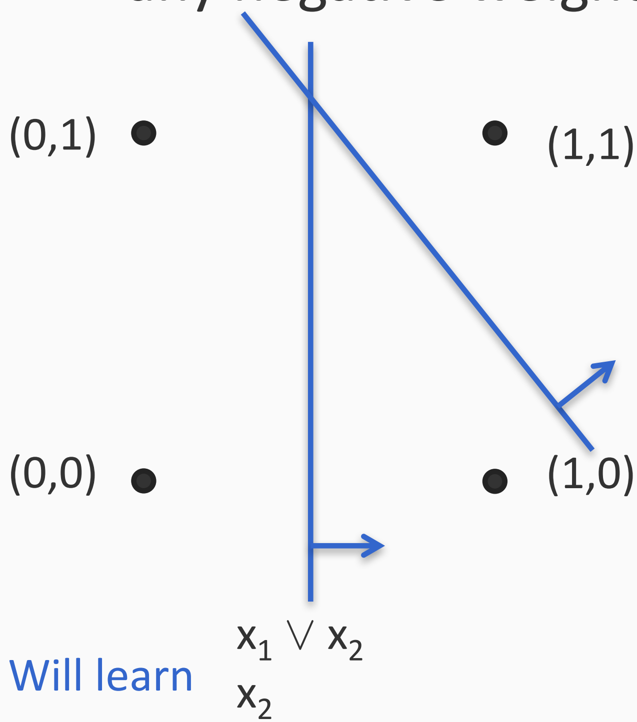
This lecture

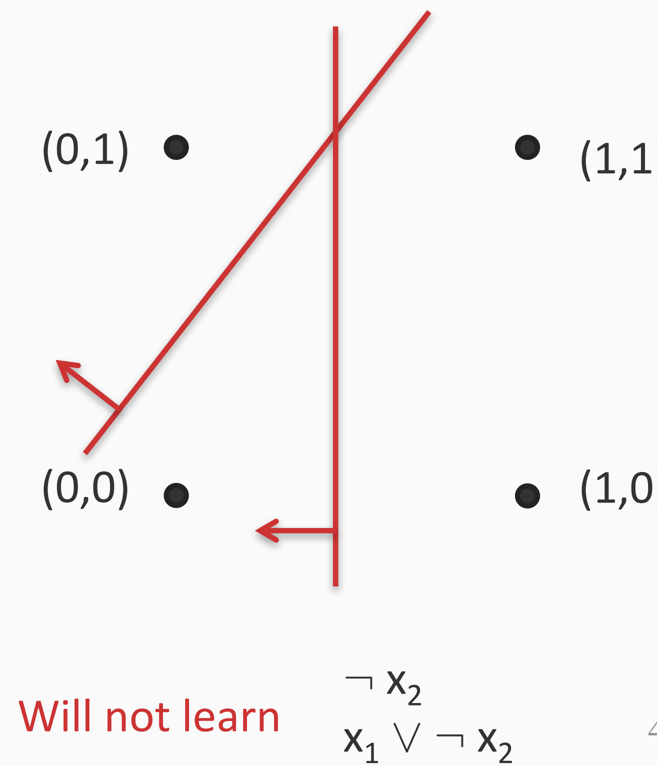
- The Winnow Algorithm
- Winnow mistake bound
- Generalizations

What can Winnow represent?

The version we saw can only learn monotone functions

- Why?
- Only multiplying and dividing the weights will never get us any negative weights





Balanced Winnow

- Duplicate the variables
  - If x^+_i represents a Boolean variable, then, introduce a new variable x^-_i to denote its negation
  - That is, learn a monotone function over the 2n variables (w^+_i for each x^+_i and w^-_i for each x^-_i)
  - Effective weight vector is the difference of the two. That is, prediction is performed as:
    - Prediction = +1 if $(\mathbf{w}^+ - \mathbf{w}^-)^T\mathbf{x} \geq \theta$, else prediction = -1
  - Modify the update rule so that whenever w_i is promoted, w^-_i should be demoted and vice versa.
- Can learn any linear threshold unit

Balanced Winnow

Given a training set D = $\{(\boldsymbol{x}, y)\}$, $\boldsymbol{x} \in \Re^n$, $y \in \{-1,1\}$

1. Initialize: $\mathbf{w}^+ = (1,1,1,1\ldots,1)$, $\mathbf{w}^- = (1,1,1,1\ldots,1) \in \Re^n$, $\theta = n$

2. For each training example $(\boldsymbol{x}, y)$:
   – Predict $y' = \mathrm{sgn}((\mathbf{w}^+ - \mathbf{w}^-)^T\boldsymbol{x} - \theta)$
   – If $y = +1$ and $y' = -1$ then:
     - Update weight $\mathbf{w}^+_i \leftarrow 2\mathbf{w}^+_i$ *only* for those features $\boldsymbol{x}_i$ that are 1
     - Update weight $\mathbf{w}^-_i \leftarrow \mathbf{w}^-_i/2$ *only* for those features $\boldsymbol{x}_i$ that are 1

   Else if $y = -1$ and $y' = +1$ then:
     - Update weight $\mathbf{w}^+_i \leftarrow \mathbf{w}^+_i/2$ *only* for those features $\boldsymbol{x}_i$ that are 1
     - Update weight $\mathbf{w}^-_i \leftarrow 2\mathbf{w}^-_i$ *only* for those features $\boldsymbol{x}_i$ that are 1

Downsides of this approach?

Perceptron and Winnow

- Both are:
  - Mistake bound algorithms
  - Learn linear threshold units
  - Are generally robust

- Which algorithm should you use??
  - Multiplicative algorithms: If you believe that the hidden target function is sparse
  - Additive algorithms: If you believe that your target function could be a dense vector
    - What if the target function is a dense vector but each example is sparse? (If time permits, we will see additive algorithms that are designed for this regime)

Summary: What Winnow so far?

- A multiplicative update algorithm
  - Learns a linear classifier when very few attributes are relevant
  - Mistake bound only weakly (logarithmically) depends on the number of attributes

- Robust to both classification and attribute noise
  - In general, instead of multiplying and dividing by 2, we could do so by $(1 + \epsilon)$ for some small ϵ