Symbolic Logic: An Introduction

THE UNIVERSITY OF UTAH

Normal forms

What is a normal form?

- Normal forms are standardized ways to write Boolean formulas
- They are universal: they can express every Boolean formula
- Different normal forms admit different algorithmic advantages

Negation normal form

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  These can be converted into NNF by

  - Replacing implications and double implications using their definitions
  - Using De Morgan's laws to push the negation inwards

Conjunctive normal form (CNF)

A conjunctive normal form is a conjunction of disjunctions of literals

$$(l_{11} \vee l_{12} \vee \cdots) \wedge (l_{21} \vee l_{22} \vee \cdots) \wedge \cdots \wedge (l_{n1} \vee l_{n2} \vee \cdots)$$

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$$(l_{11} \lor l_{12} \lor \cdots) \land (l_{21} \lor l_{22} \lor \cdots) \land \cdots \land (l_{n1} \lor l_{n2} \lor \cdots)$$

Compactly

$$\bigwedge_{j} \left(\bigvee_{i=1}^{n_j} l_{ij} \right)$$

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Validity is easy to check in a CNF (why?)

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Disjunctive normal form (DNF)

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Compactly

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Satisfiability is easy to check in a DNF (why?)

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An example illustrating normal forms

Consider the formula $A \rightarrow (B \wedge C)$

In CNF, it is $(\neg A \vee B) \wedge (\neg A \vee B)$

In DNF, it is $\neg A \vee (B \wedge C)$

Both of these are in NNFs

- The negation normal form is not unique