Symbolic Logic: An Introduction

THE UNIVERSITY OF UTAH

Propositional Logic: Semantics

Expectations from Semantics in Propositional Logic

Informal goal: What does a formula (or sentence) mean?

- The definition of semantics should specify which formulas are **`true`** (i.e., T) and which ones are **`false`** (i.e., ⊥)

Semantics should provide guidance for verification

- That is, it should guide methods for determining conditions under which a formula is true

Given a certain assignment to variables, what does a formula evaluate to?

Example

Given the assignment $\mathrm{I} = \{P_1 \mapsto \mathbf{true}, P_2 \mapsto \mathbf{false}, P_3 \mapsto \mathbf{true}\}$, what is the value of the following formulas:

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We write $I \models F$

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If a formula F evaluates to **false** under an interpretation I

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We write $I \not\models F$

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Given an interpretation, we can compute the truth value of the formula

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The interpretation is called a **satisfying interpretation** or a **model** for the formula

We write $I \vDash F$ "Interpretation I entails or models formula F"

If a formula F evaluates to **false** under an interpretation I

The interpretation is called a **falsifying interpretation** or a **counter-model** for the formula

We write $I \nvDash F$ "Interpretation I does not entail/does not model formula F"

Semantics of propositional logic

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1. Constants
   - Every interpretation entails the atom ⊤

     That is, **`true`** remains **`true`** no matter how we assign variables

$$\forall I, I \vDash \top$$

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   - Every interpretation entails the atom $\top$

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     $$\forall I, I \vDash \top$$

   - No interpretation can entail the atom $\bot$

     That is, **`false`** remains **`false`** no matter how we assign variables

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   - Recall: Interpretations explicitly assign truth values to variables
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    An interpretation I is a model for a formula $F_1 \wedge F_2$ if, and only if, the interpretation is a model for the formula F_1 and a model for the formula F_2

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$$I \vDash F_1 \leftrightarrow F_2 \quad \text{iff} \quad (I \vDash F_1 \text{ and } I \vDash F_2) \text{ or } (I \nvDash F_1 \text{ and } I \nvDash F_2)$$

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p	q	$p \rightarrow q$
true	true	true

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p	q	$p \to q$
true	**true**	**true**

In the world where both variables take the value **true**, the formula $p \to q$ takes the value **true**

But there are other worlds, where the variables take other values

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p	q	$p \rightarrow q$
true	true	true
true	false	false
false	true	true
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We can examine the truth values of any formula over all worlds with a truth table

Truth tables enumerate all possible interpretations

p	q	$p \rightarrow q$	$(p \rightarrow q) \rightarrow \neg p$	$(p \wedge q) \rightarrow p$
true	true	true	false	true
true	false	false	true	true
false	true	true	true	true
false	false	true	true	true

Truth tables enumerate all possible interpretations

p	q	$p \to q$	$(p \to q) \to \neg p$	$(p \land q) \to p$
true	**true**	**true**	**false**	**true**
true	**false**	**false**	**true**	**true**
false	**true**	**true**	**true**	**true**
false	**false**	**true**	**true**	**true**

Note that this formula is **true** in every world

Such a formula is called a *tautology*