Symbolic Logic: An Introduction

THE UNIVERSITY OF UTAH

Propositional Logic: Satisfiability & Validity

Satisfiable formulas

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The formula $p \rightarrow q$ is satisfiable

The interpretation $I = \{p \mapsto \textbf{true}, q \mapsto \textbf{true}\}$
is a model for the formula

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A formula F is said to be **satisfiable** if there exists some interpretation I such that $\mathrm{I} \vDash F$

If no such interpretation exists, then the formula is said to be **unsatisfiable**

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The formula $p \leftrightarrow \neg p$ is **unsatisfiable**

Every interpretation (there are only two) assigns value $\mathbf{false}$ to the formula

Valid formulas

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- That is, for any interpretation I, we have $\mathrm{I} \vDash F$
- Valid formulas are tautologies

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One approach: Write out the truth table

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p	q
true	true
true	false
false	true
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true	false	true
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true	false	true	true	true
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Satisfiability & Validity

There a duality between satisfiability and validity

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Every interpretation entails the formula

No interpretation that entails it

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Why? The proof is easy

We need to prove the "if" part and the "only if" part separately

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The "only if" part of the proof

If a formula F is unsatisfiable, then…
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That is, every interpretation is a model for $\neg F$.
That is, the negation $\neg F$ is valid.

Logical Entailment

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We say that F entails α, or that α is a logical consequence of F, if there is no interpretation where F is **`true`** and α is **`false`**

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Some examples of logical entailment:

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P_1 \rightarrow P_2 \quad &\vDash \quad (P_1 \land P_3) \rightarrow P_2 \\
\big((P_1 \land P_2) \rightarrow P_3\big) \rightarrow P_4 \quad &\vDash \quad (P_1 \land P_3) \rightarrow P_4
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Key takeaway

It is enough if we have algorithms for satisfiability. Other properties can follow from them