Support Vector Machines: Training with Stochastic Gradient Descent

Machine Learning

THE UNIVERSITY OF UTAH
Support vector machines

• Training by maximizing margin

• The SVM objective

• **Solving the SVM optimization problem**

• Support vectors, duals and kernels
SVM objective function

\[
\min_{\mathbf{w}} \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \sum_i \max(0, 1 - y_i \mathbf{w}^T \mathbf{x})
\]

**Regularization term:**
- Maximize the margin
- Imposes a preference over the hypothesis space and pushes for better generalization
- Can be replaced with other regularization terms which impose other preferences

**Empirical Loss:**
- Hinge loss
- Penalizes weight vectors that make mistakes
- Can be replaced with other loss functions which impose other preferences

A hyper-parameter that controls the tradeoff between a large margin and a small hinge-loss
Outline: Training SVM by optimization

1. Review of convex functions and gradient descent
2. Stochastic gradient descent
3. Gradient descent vs stochastic gradient descent
4. Sub-derivatives of the hinge loss
5. Stochastic sub-gradient descent for SVM
6. Comparison to perceptron
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Solving the SVM optimization problem

\[
\min_{\mathbf{w}} \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \sum_i \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i)
\]

This function is **convex** in \( \mathbf{w} \)
A function $f$ is **convex** if for every $u, v$ in the domain, and for every $\lambda \in [0,1]$ we have

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v)$$
A function $f$ is **convex** if for every $u, v$ in the domain, and for every $\lambda \in [0,1]$ we have

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v)$$

From geometric perspective

Every tangent plane lies below the function
Convex functions

Some ways to show that a function is convex:

1. Using the definition of convexity
2. Showing that the second derivative is positive (for one dimensional functions)
3. Showing that the second derivative is positive semi-definite (for vector functions)
Not all functions are convex

These are concave

These are neither

\[ f(\lambda u + (1 - \lambda)v) \geq \lambda f(u) + (1 - \lambda)f(v) \]
Convex functions are convenient

A function $f$ is *convex* if for every $u, v$ in the domain, and for every $\lambda \in [0,1]$ we have

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v)$$

In general: Necessary condition for $x$ to be a minimum for the function $f$ is that the gradient $\nabla f(x) = 0$

For convex functions, this is both necessary *and* sufficient
Solving the SVM optimization problem

\[
\min_w \frac{1}{2} w^T w + C \sum_i \max(0, 1 - y_i w^T x_i)
\]

This function is convex in \( w \)

- This is a quadratic optimization problem because the objective is quadratic
- Older methods: Used techniques from Quadratic Programming
  - Very slow
- No constraints, can use \textit{gradient descent}
  - Still very slow!
Gradient descent

General strategy for minimizing a function $J(w)$

- Start with an initial guess for $w$, say $w^0$
- Iterate till convergence:
  - Compute the gradient of the gradient of $J$ at $w^t$
  - Update $w^t$ to get $w^{t+1}$ by taking a step in the opposite direction of the gradient

We are trying to minimize

$$J(w) = \min_w \frac{1}{2} w^T w + C \sum_i \max(0, 1 - y_i w^T x_i)$$

**Intuition**: The gradient is the direction of steepest increase in the function. To get to the minimum, go in the opposite direction.
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Intuition: The gradient is the direction of steepest increase in the function. To get to the minimum, go in the opposite direction
Gradient descent for SVM

1. Initialize $\mathbf{w}^0$

2. For $t = 0, 1, 2, \ldots$
   1. Compute gradient of $J(\mathbf{w})$ at $\mathbf{w}^t$. Call it $\nabla J(\mathbf{w}^{t+1})$
   2. Update $\mathbf{w}$ as follows:
      $$ \mathbf{w}^{t+1} \leftarrow \mathbf{w}^t - r \nabla J(\mathbf{w}^t) $$

$r$: The learning rate.

We are trying to minimize

$$ J(\mathbf{w}) = \min_w \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \sum_i \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i) $$
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- Review of convex functions and gradient descent

2. Stochastic gradient descent

3. Gradient descent vs stochastic gradient descent

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   1. Compute gradient of $J(\mathbf{w})$ at $\mathbf{w}^t$. Call it $\nabla J(\mathbf{w}^{t+1})$

Gradient of the SVM objective requires summing over the entire training set

**Slow**, does not really scale

$r$: Called the learning rate

$J(\mathbf{w}) = \min_{\mathbf{w}} \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \sum_i \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i)$
Stochastic gradient descent for SVM

Given a training set $S = \{(x_i, y_i)\}, \ x \in \mathbb{R}^d, y \in \{-1, 1\}$

1. Initialize $w^0 = 0 \in \mathbb{R}^d$
2. For epoch $= 1 \ldots T$:
   1. Pick a random example $((x_i, y_i))$ from the training set $S$
   2. Treat $((x_i, y_i))$ as a full dataset and take the derivative of the SVM objective, $J$, at the current $w$
   3. Update: $w^{!} \leftarrow w^{!} - \gamma \nabla J(w)$

3. Return final $w$
Stochastic gradient descent for SVM

Given a training set $S = \{(x_i, y_i)\}$, $x \in \mathbb{R}^d$, $y \in \{-1, 1\}$

1. Initialize $w^0 = 0 \in \mathbb{R}^d$
2. For epoch = 1 ... $T$:
   1. Pick a random example $(x_i, y_i)$ from the training set $S$

\[ J(w) = \frac{1}{2} w^T w + C \sum_i \max(0, 1 - y_i w^T x_i) \]

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   1. Pick a random example $(x_i, y_i)$ from the training set $S$
   2. Treat $(x_i, y_i)$ as a full dataset and take the derivative of the SVM objective $\hat{J}$ at the current $w^{t-1}$. Call it $\nabla \hat{J}(w^{t-1})$

3. Return final $w$
Stochastic gradient descent for SVM

Given a training set \( S = \{(x_i, y_i)\}, \ x \in \mathbb{R}^d, \ y \in \{-1,1\} \)

1. Initialize \( w^0 = 0 \in \mathbb{R}^d \)

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   2. Treat \((x_i, y_i)\) as a full dataset and take the derivative of the SVM objective \( \hat{J} \) at the current \( w^{t-1} \). Call it \( \nabla \hat{J}(w^{t-1}) \)

\[
\hat{J}(w) = \min_w \frac{1}{2} w^T w + C \max(0, 1 - y_i w^T x_i)
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$$\hat{J}(w) = \min_w \frac{1}{2} w^T w + C \sum_i \max(0, 1 - y_i w^T x_i)$$

3. Update: $w^t \leftarrow w^{t-1} - \gamma_t \nabla \hat{J}(w^{t-1})$

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Stochastic gradient descent for SVM

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   3. Update: $w^t \leftarrow w^{t-1} - \gamma_t \nabla J(w^{t-1})$

3. Return final $w$

This algorithm is guaranteed to converge to the minimum of $J$ if $\gamma_t$ is small enough.
Why? The objective $J(w)$ is a **convex** function.
Outline: Training SVM by optimization

- Review of convex functions and gradient descent
- Stochastic gradient descent

3. **Gradient descent vs stochastic gradient descent**

4. Sub-derivatives of the hinge loss

5. Stochastic sub-gradient descent for SVM

6. Comparison to perceptron
Gradient Descent vs SGD

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Stochastic Gradient descent
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Many more updates than gradient descent, but each individual update is less computationally expensive.
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Stochastic gradient descent for SVM

Given a training set $S = \{(x_i, y_i)\}, \ x \in \mathbb{R}^d, y \in \{-1, 1\}$

1. Initialize $w^0 = 0 \in \mathbb{R}^d$

2. For epoch = 1 ... T:
   1. Pick a random example $(x_i, y_i)$ from the training set $S$
   
   2. Treat $(x_i, y_i)$ as a full dataset and take the derivative of the SVM objective at the current $w^{t-1}$ to be $\nabla J(w^{t-1})$

3. Update: $w^t \leftarrow w^{t-1} - \gamma_t \nabla J(w^{t-1})$

3. Return final $w$

\[
J(w) = \min_w \frac{1}{2} w^T w + C \sum_i \max(0, 1 - y_i w^T x)
\]
Hinge loss is not differentiable!

What is the derivative of the hinge loss with respect to \( w \)?

\[
J(w) = \min_w \frac{1}{2} w^T w + C \max(0, 1 - y_i w^T x_i)
\]
Detour: Sub-Gradients

Generalization of gradients to non-differentiable functions

(Remember that every tangent is a hyperplane that lies below the function for convex functions)

Informally, a sub-tangent at a point is any hyperplane that lies below the function at the point.

A sub-gradient is the slope of that line.
Sub-gradients

Formally, a vector $g$ is a subgradient to $f$ at point $x$ if

$$f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y$$
Sub-Gradients

Formally, a vector \( g \) is a subgradient to \( f \) at point \( x \) if

\[
f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y
\]

\( f \) is differentiable at \( x_1 \)
Tangent at this point

\[
f(x_1) + g_1^T(x - x_1)
\]

\( g_1 \) is a gradient at \( x_1 \)
Sub-Gradients

Formally, a vector $g$ is a subgradient to $f$ at point $x$ if

$$f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y$$

$f$ is differentiable at $x_1$
Tangent at this point

$$f(x_1) + g_1^T(x - x_1)$$

$g_1$ is a gradient at $x_1$

$g_2$ and $g_3$ is are both subgradients at $x_2$

[Example from Boyd]
Sub-gradient of the SVM objective

\[ J^t(w) = \frac{1}{2} w^T w + C \max (0, 1 - y_i w^T x_i) \]

**General strategy:** First solve the max and compute the gradient for each case.
Sub-gradient of the SVM objective

\[ J^t(w) = \frac{1}{2} w^T w + C \max(0, 1 - y_i w^T x_i) \]

**General strategy**: First solve the max and compute the gradient for each case.

\[ \nabla J^t = \begin{cases} 
  w & \text{if } \max(0, 1 - y_i w^T x_i) = 0 \\
  w - C y_i x_i & \text{otherwise}
\end{cases} \]
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\end{cases} \]

Given a training set \( S = \{(x_i, y_i)\}, \ x \in \mathbb{R}^d, y \in \{-1, 1\} \)

1. Initialize \( w = 0 \in \mathbb{R}^d \)

3. Return \( w \)
Stochastic **sub-gradient** descent for SVM

\[
\nabla J^t = \begin{cases} 
  w & \text{if } \max(0, 1 - y_i w^T x_i) = 0 \\
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\end{cases}
\]

Given a training set \( S = \{(x_i, y_i)\} \), \( x \in \mathbb{R}^d, y \in \{-1, 1\} \)

1. Initialize \( w = 0 \in \mathbb{R}^d \)

2. For epoch = 1 ... T:

3. Return \( w \)
Stochastic sub-gradient descent for SVM

Given a training set \( S = \{ (\mathbf{x}_i, y_i) \} \), \( \mathbf{x} \in \mathbb{R}^d \), \( y \in \{-1, 1\} \)

1. Initialize \( \mathbf{w} = 0 \in \mathbb{R}^d \)

2. For epoch = 1 ... T:
   
   For each training example \( (\mathbf{x}_i, y_i) \in S \):

   Update \( \mathbf{w} \leftarrow \mathbf{w} - \gamma_t \nabla J \)

3. Return \( \mathbf{w} \)

\[
\nabla J^t = \begin{cases} 
\mathbf{w} & \text{if } \max (0, 1 - y_i \mathbf{w}^T \mathbf{x}_i) = 0 \\
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Stochastic sub-gradient descent for SVM

\[ \nabla J^t = \begin{cases} 
    w & \text{if } \max(0, 1 - y_i w^T x_i) = 0 \\
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Given a training set \( S = \{(x_i, y_i)\}, \quad x \in \mathbb{R}^d, \quad y \in \{-1, 1\} \)

1. Initialize \( w = 0 \in \mathbb{R}^d \)
2. For epoch = 1 ... T:
   
   For each training example \( (x_i, y_i) \in S \):
   
   If \( y_i w^T x_i \leq 1 \):
   
   \[ w \leftarrow (1 - \gamma_t)w + \gamma_t C y_i x_i \]
   
   else:
   
   \[ w \leftarrow (1 - \gamma_t)w \]

3. Return \( w \)
Stochastic sub-gradient descent for SVM

Given a training set \( S = \{(x_i, y_i)\}, \ x \in \mathbb{R}^d, y \in \{-1, 1\} \)

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2. For epoch = 1 \( \ldots \) T:
   For each training example \( (x_i, y_i) \in S \):
   
   If \( y_i \mathbf{w}^T x_i \leq 1 \):
   \[
   \mathbf{w} \leftarrow (1 - \gamma_t) \mathbf{w} + \gamma_t C y_i x_i
   \]
   else:
   \[
   \mathbf{w} \leftarrow (1 - \gamma_t) \mathbf{w}
   \]

3. Return \( \mathbf{w} \)

\( \gamma_t \): learning rate, many tweaks possible
Stochastic sub-gradient descent for SVM

Given a training set \( S = \{(x_i, y_i)\}, \quad x \in \mathbb{R}^d, \quad y \in \{-1, 1\}\)

1. Initialize \( w = 0 \in \mathbb{R}^d \)

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   For each training example \( (x_i, y_i) \in S \):
   
   If \( y_i w^T x_i \leq 1 \):
   
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   w \leftarrow (1 - \gamma_t)w + \gamma_t C y_i x_i
   \]
   
   else:
   
   \[
   w \leftarrow (1 - \gamma_t)w
   \]

3. Return \( w \)

\( \gamma_t \): learning rate, many tweaks possible

Important to shuffle examples at the start of each epoch
Convergence and learning rates

With enough iterations, it will converge in expectation

Provided the step sizes are “square summable, but not summable”

• Step sizes $\gamma_t$ are positive
• Sum of squares of step sizes over $t = 1$ to $1$ is not infinite
• Sum of step sizes over $t = 1$ to $1$ is infinity

• Some examples: $\gamma_t = \frac{\gamma_0}{1 + \frac{\gamma_0 t}{C}}$ or $\gamma_t = \frac{\gamma_0}{1 + t}$
Convergence and learning rates

• Number of iterations to get to accuracy within $\epsilon$

• For strongly convex functions, N examples, d dimensional:
  - Gradient descent: $O\left( Nd \ln \frac{1}{\epsilon} \right)$
  - Stochastic gradient descent: $O\left( \frac{d}{\epsilon} \right)$

• More subtleties involved, but SGD is generally preferable when the data size is huge
Convergence and learning rates

- Number of iterations to get to accuracy within $\varepsilon$

- For strongly convex functions, $N$ examples, $d$ dimensional:
  - Gradient descent: $O \left( N d \ln \frac{1}{\varepsilon} \right)$
  - Stochastic gradient descent: $O \left( \frac{d}{\varepsilon} \right)$

- More subtleties involved, but SGD is generally preferable when the data size is huge

- Recently, many variants that are based on this general strategy
  - Examples: Adagrad, momentum, Nesterov’s accelerated gradient, Adam, RMSProp, etc...
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Stochastic sub-gradient descent for SVM

Given a training set \( S = \{(x_i, y_i)\}, \ x \in \mathbb{R}^d, y \in \{-1,1\} \)

1. Initialize \( w = 0 \in \mathbb{R}^d \)

2. For epoch = 1 ... T:
   
   For each training example \((x_i, y_i) \in S:\)
   
   If \( y_i w^T x_i \leq 1: \)
   
   \[
   w \leftarrow (1 - \gamma_t)w + \gamma_tC y_i x_i
   \]

   else:

   \[
   w \leftarrow (1 - \gamma_t)w
   \]

3. Return \( w \)

Compare with the Perceptron update:

If \( y_i w^T x_i \leq 0, \) update \( w \leftarrow w + \gammaTy_i x_i \)
Perceptron vs. SVM

- Perceptron: Stochastic sub-gradient descent for a different loss
  - No regularization though

\[ L_{Perceptron}(y, \mathbf{x}, \mathbf{w}) = \max(0, -y\mathbf{w}^T\mathbf{x}) \]

- SVM optimizes the hinge loss
  - With regularization

\[ L_{Hinge}(y, \mathbf{x}, \mathbf{w}) = \max(0, 1 - y\mathbf{w}^T\mathbf{x}) \]
SVM summary from optimization perspective

• Minimize regularized hinge loss

• Solve using stochastic gradient descent
  – Very fast, run time does not depend on number of examples

  – Compare with Perceptron algorithm: Perceptron does not maximize margin width
    • Perceptron variants can force a margin

  – Convergence criterion is an issue; can be too aggressive in the beginning and get to a reasonably good solution fast; but convergence is slow for very accurate weight vector

• Other successful optimization algorithms exist
  – Eg: Dual coordinate descent, implemented in liblinear

Questions?