Online Learning

Machine Learning

THE UNIVERSITY OF UTAH

Some slides based on lectures from Dan Roth, Avrim Blum and others

Big picture

Big picture

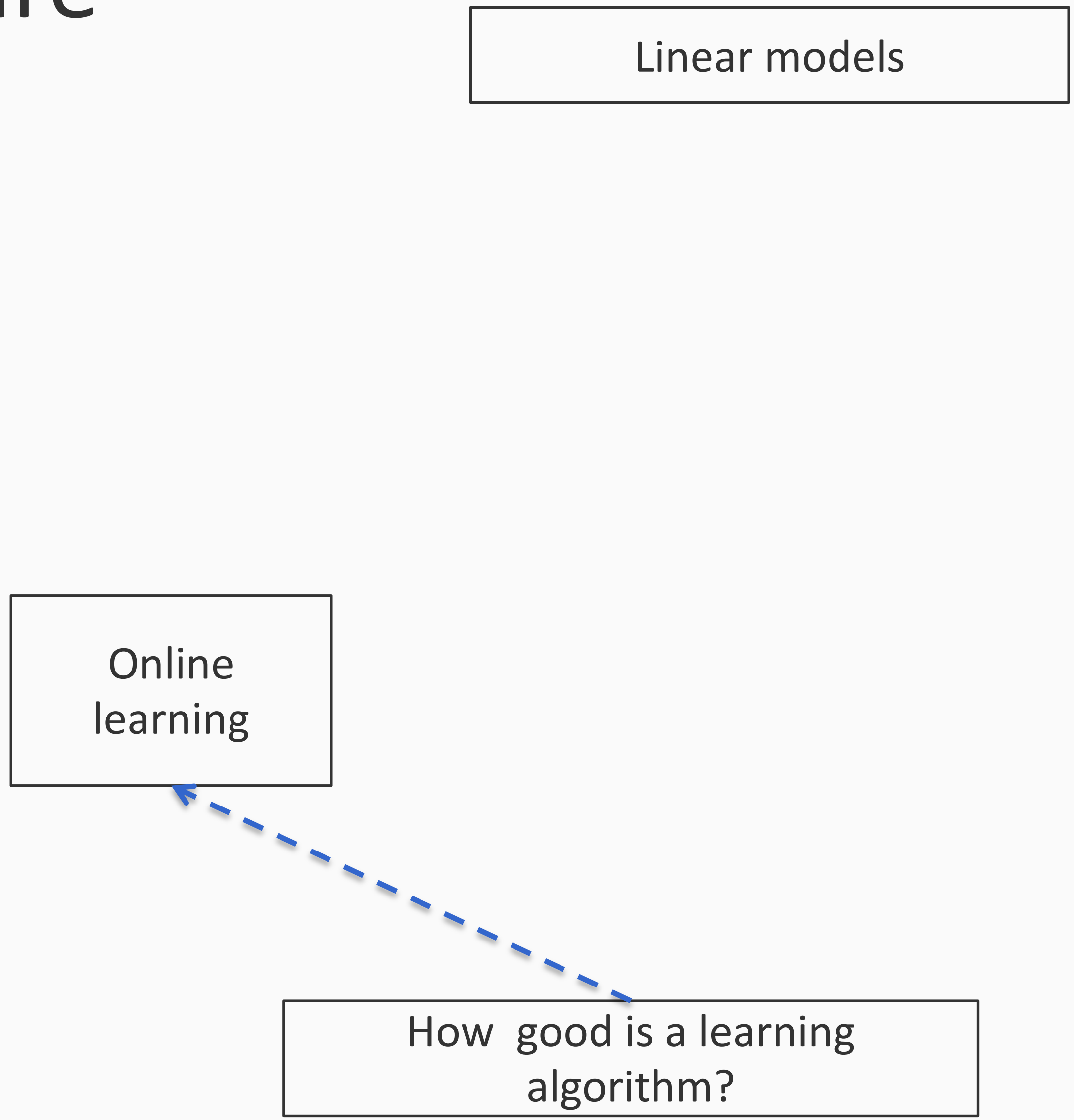
Last lecture: Linear models

Big picture

Linear models

How good is a learning algorithm?

Big picture



Linear models
Online learning
How good is a learning algorithm?

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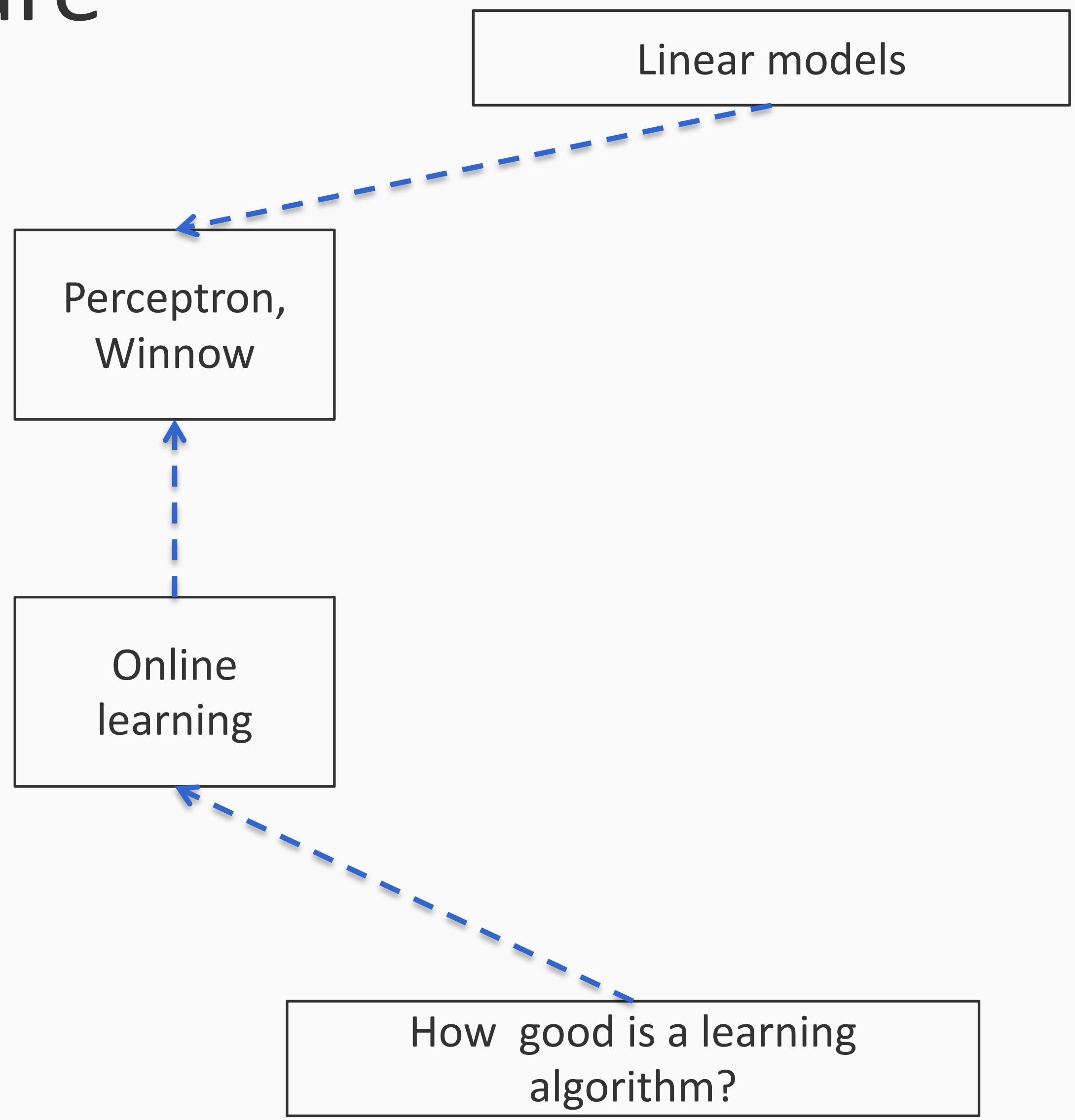
Linear models
Perceptron, Winnow
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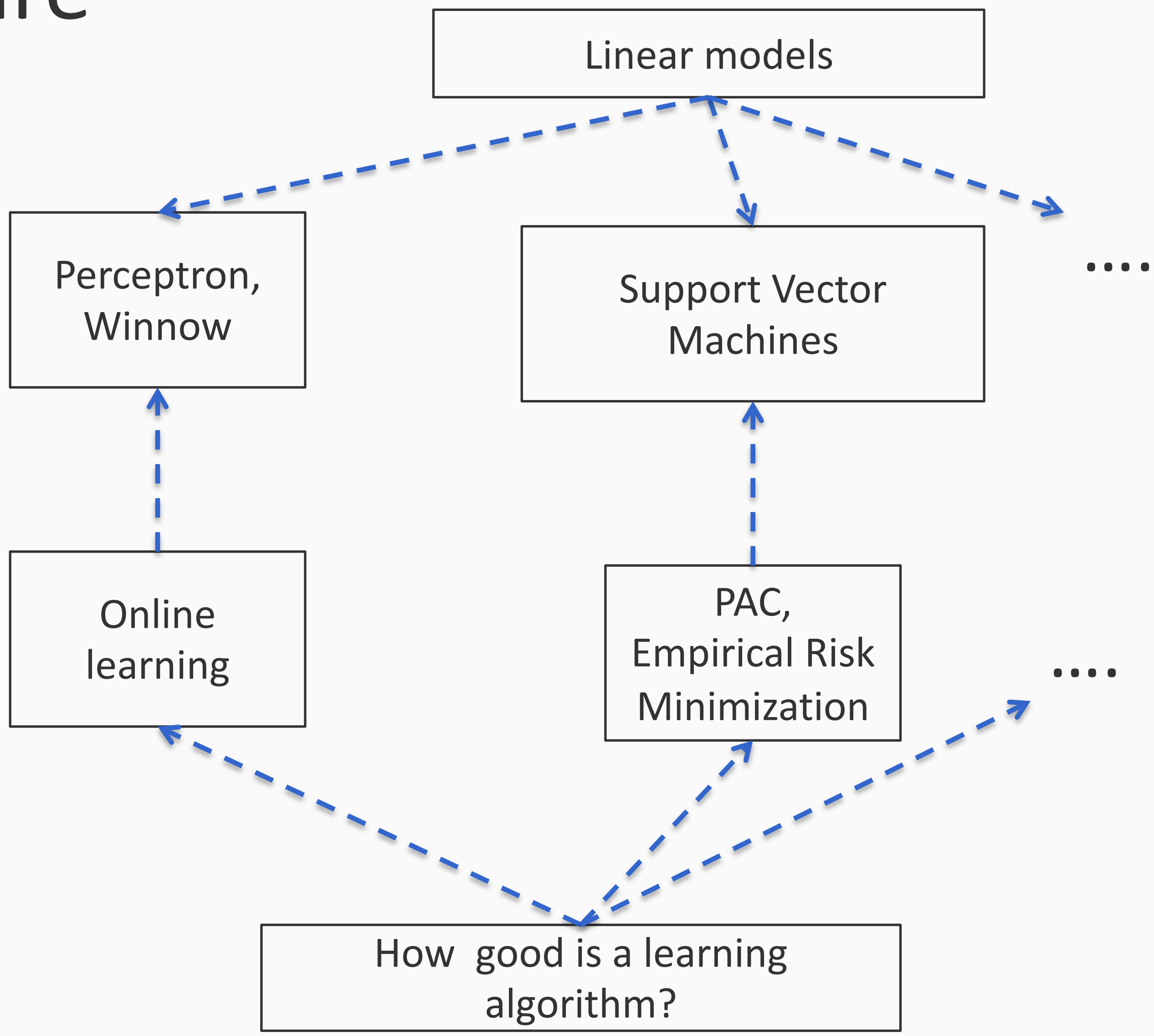
Linear models
Perceptron, Winnow
Support Vector Machines
....
Online learning
PAC, Empirical Risk Minimization
....
How good is a learning algorithm?

Mistake bound learning

- The mistake bound model
- A proof of concept mistake bound algorithm: The Halving algorithm
- Examples
- Representations and ease of learning

Coming up…

- Mistake-driven learning

- Learning algorithms for learning a linear function over the feature space
  - Perceptron (with many variants)
  - General Gradient Descent view

Issues to watch out for

- Importance of Representation
- Complexity of Learning
- More about features

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Motivation

Consider a learning problem in a very high dimensional space

$$[x_1, x_2, \cdots, x_{1000000}]$$

And assume that the function space is very sparse (the function of interest depends on a small number of attributes.)

$$f = x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_{100}$$

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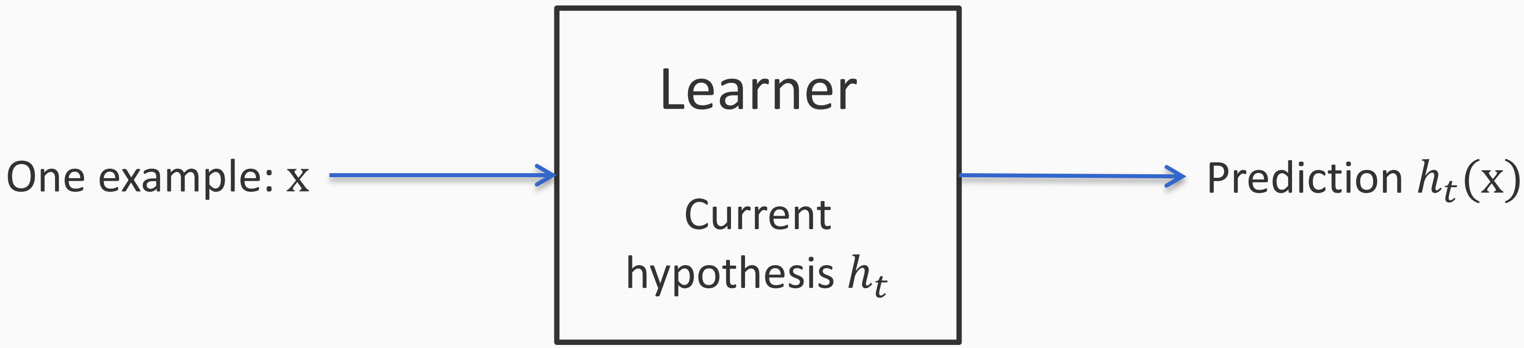
- Can we develop an algorithm that depends only *weakly* on the dimensionality and mostly on the number of relevant attributes?
- How should we represent the hypothesis?

An illustration of mistake driven learning



Loop forever:

1. Receive example x

2. Make a prediction using the current hypothesis $h_t(\mathrm{x})$

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    - Update h_t to h_{t+1}

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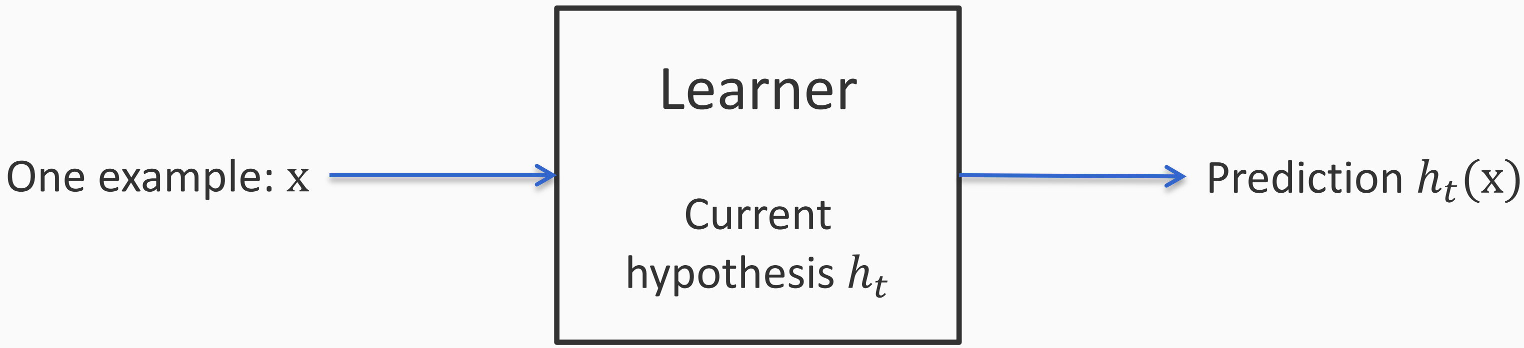

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Only need to define how prediction and update behave

Can such a simple scheme work? How do we quantify what "work" means?

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  - Target f: $\mathcal{X} \to \{0,1\}$, $f \in C$ the concept class (parameterized by n)

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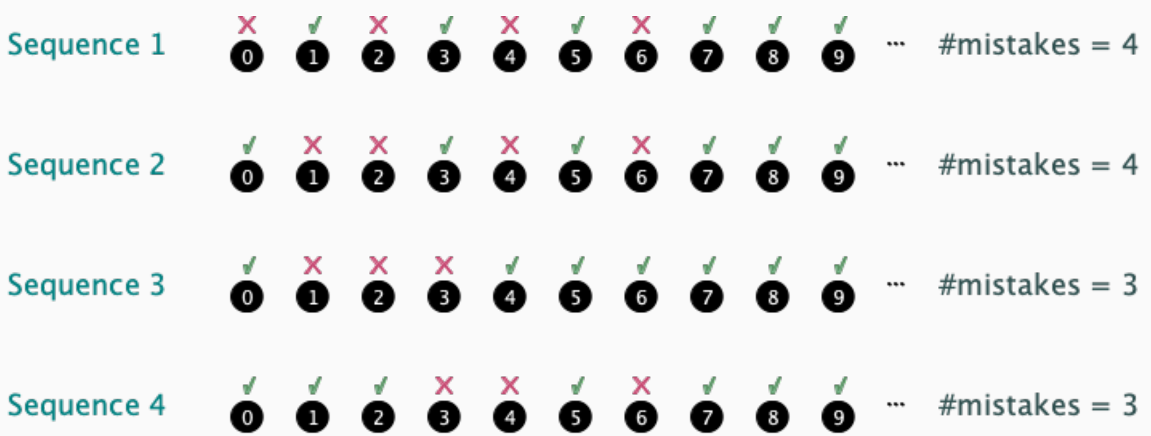

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Learnability in the mistake bound model

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Learnability in the mistake bound model

- Not the most general setting for online learning
- Not the most general metric
- Other metrics: Regret, cumulative loss

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Online Learning

- No assumptions about the distribution of examples

- Examples are presented to the learning algorithm in a sequence. *Could be adversarial!*

  For each example:

  1. Learner gets an unlabeled example
  2. Learner makes a prediction
  3. Then, the true label is revealed

- In the mistake bound model, we only count the number of mistakes

Online Learning

- Simple and intuitive model, widely applicable

- Important in the case of very large data sets, when the data cannot fit memory (streaming data)

- Evaluation: We will try to make the smallest number of mistakes in the long run.

  – Some things to think about:
    - What is the relation to the “real” goal? What is the real goal of learning?
    - Does online learning generate a hypothesis that does well on previously unseen data?

Online/Mistake Bound Learning

- No notion of data distribution; a worst case model

- No (or not much) memory: get example → update hypothesis → get rid of it

- Drawbacks:
  - Too simple
  - Global behavior: not clear when will the mistakes be made

- Advantages:
  - Simple
  - Many issues arise already in this setting
  - Generic conversion to other learning models (online-to-batch conversion)

Is counting mistakes enough?

- Under the mistake bound model, we are not concerned about the number of examples needed to learn a function

- We only care about not making mistakes

- Eg: Suppose the learner is presented the *same example* over and over
  - *Under the mistake bound model, it is okay*
  - *We won't be able to learn the function, but we won't make any mistakes either!*

Mistake bound learning

- The mistake bound model
- A proof of concept mistake bound algorithm: The Halving algorithm
- Examples
- Representations and ease of learning

Can mistake bound algorithms exist?

Before getting to a more useful mistake bound algorithm, let's see a proof-of-concept mistake bound algorithm

The Halving algorithm

Generic Mistake Bound Algorithms

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Is this a mistake bound algorithm? Depends on what C is

Can we do better than CON?

The Halving Algorithm

- Let C be a finite concept class
- Goal: Learn $f \in C$

- Initialize $C_0 = C$, the set of all possible functions

We will construct a series of sets of functions C_i

- Learning ends when there is only one element in C_i

The Halving Algorithm

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Suppose it makes n mistakes. Finally, we will have the final set of concepts C_n with one element

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Questions?

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- Hard to compute
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For the most difficult concept in the class,

for the most difficult sequence of examples,

the optimal mistake bound algorithm makes the fewest number of mistakes

The Halving Algorithm

- Hard to compute

- In some concept classes, Halving is *optimal*
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- In general, to be optimal, instead of guessing in accordance with the majority of the valid concepts, we should guess according to the concept group that gives the least number of expected mistakes (even harder to compute)

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for the most difficult sequence of examples,

the optimal mistake bound algorithm makes the fewest number of mistakes

Summary: The Halving algorithm

- A simple algorithm for *finite* concept spaces
  - Stores a set of hypotheses that it iteratively refines
    - Receive an input
    - Prediction: the label of the majority of hypotheses still under consideration
    - Update: If incorrect, remove all inconsistent hypotheses
- Makes O(log|C|) mistakes for a concept class C
- Not always optimal because we care about minimizing the number of mistakes in the future!
  - What if, instead of eliminating functions that disagree with this example, we down-weight them
  - Perhaps via multiplicative or additive updates…

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Learning Conjunctions

Hidden function: conjunctions

- The learner is to learn functions like $f = x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_{100}$

- Number of conjunctions with n variables = $|\mathrm{C}| = ???$

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- There is a practical algorithm that can achieve this bound
  - Elimination: Learn from positive examples by eliminating inactive literals.

The Halving algorithm is not efficient.

Elimination is an efficient algorithm that realizes the mistake bound of the Halving algorithm

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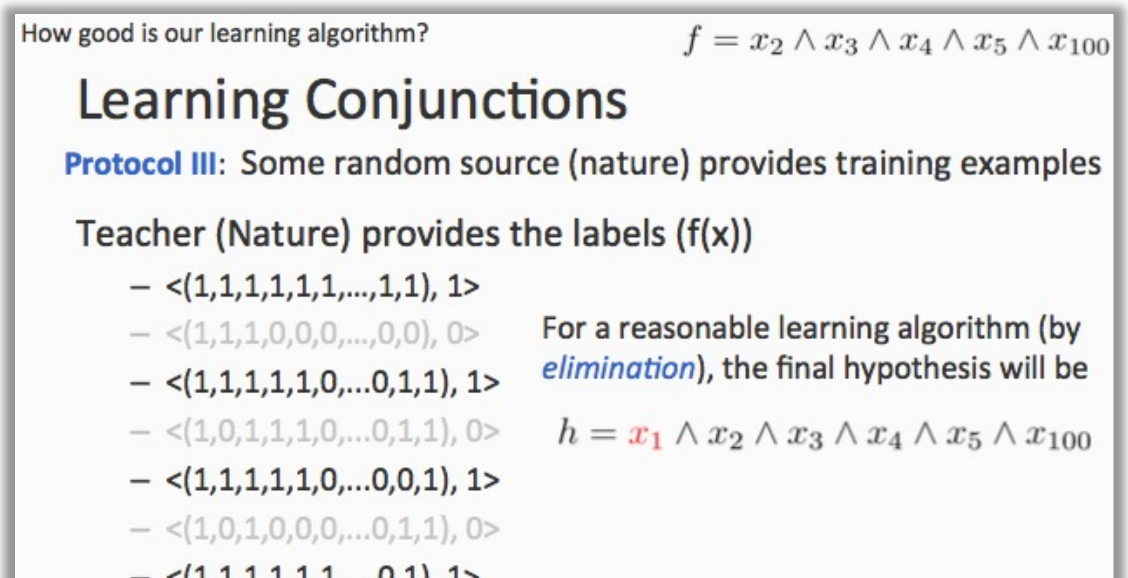

Learning Conjunctions: Elimination

$$f = x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_{100}$$

Teacher (Nature) provides the labels (f(x))

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Notation: <example, label>

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Look for the variables that are present in *every* positive example.

All other variables can be eliminated

Why?

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Whenever the output is 1, x_1 is present

$$f = x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_{100}$$

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Teacher (Nature) provides the labels (f(x))

- <(1,1,1,1,1,1,…,1,1), 1>
- <(1,1,1,0,0,0,…,0,0), 0>
- <(1,1,1,1,1,0,...0,1,1), 1>
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For a reasonable learning algorithm (by *elimination*), the final hypothesis will be

$$h = x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_{100}$$

Whenever the output is 1, x_1 is present

With the given data, we only learned an *approximation* to the true concept.
Is it good enough?

Learning Conjunctions

Hidden function: conjunctions

- The learner is to learn functions like $f = x_2 \land x_3 \land x_4 \land x_5 \land x_{100}$

- Number of conjunctions with n variables = $|C| = 3^n$
  - $\log|C| = O(n)$
- The elimination algorithm makes at most n mistakes
  - Learn from positive examples; eliminate inactive literals.

Hidden function: *k-conjunctions*

- Assume that only k<<n attributes occur in the conjunction

- Number of k-conjunctions = $2^k \binom{n}{k} \approx 2^k n^k$ Why?
  - $\log|C| = O(k \log n)$
  - <u>*Can we learn efficiently with this number of mistakes ?*</u>

Mistake bound learning

- The mistake bound model
- A proof of concept mistake bound algorithm: The Halving algorithm
- Examples
- Representations and ease of learning

Representation and efficient learning

- Assume that you want to learn conjunctions. Should your hypothesis space be the class of conjunctions?

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  $\Rightarrow$ We cannot learn the concept efficiently **as a (dis/con)junction**
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In a more expressive class, the search for a good hypothesis sometimes becomes combinatorially easier

What you should know

- What is the mistake bound model?
- Simple *proof-of-concept* mistake bound algorithms
  - CON: Makes O(|C|) mistakes
  - The Halving algorithm
    - Can learn a concept with at most log(|C|) mistakes
    - Sadly, for non-trivial functions, only useful if we don't care about storage or computation time
    - How to apply this bound to simple function classes
- Even for simple Boolean functions (conjunctions and disjunctions), learning them as linear threshold units is computationally easier