Support Vector Machines: Training with Stochastic Gradient Descent
Support vector machines

• Training by maximizing margin

• The SVM objective

• **Solving the SVM optimization problem**

• Support vectors, duals and kernels
SVM objective function

\[
\min_w \frac{1}{2} w^T w + C \sum_i \max(0, 1 - y_i w^T x_i)
\]

**Regularization term:**
- Maximize the margin
- Imposes a preference over the hypothesis space and pushes for better generalization
- Can be replaced with other regularization terms which impose other preferences

**Empirical Loss:**
- Hinge loss
- Penalizes weight vectors that make mistakes
- Can be replaced with other loss functions which impose other preferences

A **hyper-parameter** that controls the tradeoff between a large margin and a small hinge-loss
Outline: Training SVM by optimization

1. Review of convex functions and gradient descent

2. Stochastic gradient descent

3. Gradient descent vs stochastic gradient descent

4. Sub-derivatives of the hinge loss

5. Stochastic sub-gradient descent for SVM

6. Comparison to perceptron
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Solving the SVM optimization problem

\[
\min_w \frac{1}{2} w^T w + C \sum_i \max(0, 1 - y_i w^T x_i)
\]

This function is convex in \( w \)
Recall: Convex functions

A function $f$ is **convex** if for every $u, v$ in the domain, and for every $\lambda \in [0,1]$ we have

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v)$$
Recall: Convex functions

A function $f$ is **convex** if for every $u, v$ in the domain, and for every $\lambda \in [0,1]$ we have

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From geometric perspective

Every tangent plane lies below the function
Convex functions

Some ways to show that a function is convex:

1. Using the definition of convexity
2. Showing that the second derivative is positive (for one dimensional functions)
3. Showing that the second derivative is positive semi-definite (for vector functions)
Not all functions are convex

These are concave

\[ f(\lambda u + (1 - \lambda)v) \geq \lambda f(u) + (1 - \lambda)f(v) \]

These are neither
A function $f$ is **convex** if for every $u, v$ in the domain, and for every $\lambda \in [0,1]$ we have

$$f(\lambda u + (1 - \lambda)v) \leq \lambda f(u) + (1 - \lambda)f(v)$$

In general: Necessary condition for $x$ to be a minimum for the function $f$ is $\nabla f(x) = 0$

For convex functions, this is both necessary *and* sufficient
Solving the SVM optimization problem

$$\min_w \frac{1}{2} w^T w + C \sum_i \max(0, 1 - y_i w^T x_i)$$

This function is convex in $w$

- This is a quadratic optimization problem because the objective is quadratic

- Older methods: Used techniques from Quadratic Programming
  - Very slow

- No constraints, can use *gradient descent*
  - Still very slow!
Gradient descent

General strategy for minimizing a function $J(w)$

• Start with an initial guess for $w$, say $w^0$

• Iterate till convergence:
  – Compute the gradient of the gradient of $J$ at $w^t$
  – Update $w^t$ to get $w^{t+1}$ by taking a step in the opposite direction of the gradient

We are trying to minimize

$$J(w) = \frac{1}{2} w^T w + C \sum_i \max(0, 1 - y_i w^T x_i)$$

**Intuition**: The gradient is the direction of steepest increase in the function. To get to the minimum, go in the opposite direction.
Gradient descent

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Intuition: The gradient is the direction of steepest increase in the function. To get to the minimum, go in the opposite direction.
Gradient descent for SVM

1. Initialize $\mathbf{w}^0$

2. For $t = 0, 1, 2, \ldots$
   
   1. Compute gradient of $J(\mathbf{w})$ at $\mathbf{w}^t$. Call it $\nabla J(\mathbf{w}^t)$

   2. Update $\mathbf{w}$ as follows:

   $$\mathbf{w}^{t+1} = \mathbf{w}^t - r \nabla J(\mathbf{w}^t)$$

   $r$: Called the learning rate.

We are trying to minimize

$$J(\mathbf{w}) = \frac{1}{2} \mathbf{w}^T \mathbf{w} + C \sum_i \max(0, 1 - y_i \mathbf{w}^T \mathbf{x}_i)$$
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2. **Stochastic gradient descent**

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Gradient descent for SVM

1. Initialize $w^0$

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We are trying to minimize

$$J(w) = \frac{1}{2} w^T w + C \sum_i \max(0, 1 - y_i w^T x_i)$$

Gradient of the SVM objective requires summing over the entire training set

- Slow, does not really scale

$r$: Called the learning rate
Stochastic gradient descent for SVM

Given a training set $S = \{(x_i, y_i)\}$, $x \in \mathbb{R}^n$, $y \in \{-1, 1\}$

1. Initialize $w^0 = 0 \in \mathbb{R}^n$
2. For epoch $= 1 \ldots T$:

3. Return final $w$
Stochastic gradient descent for SVM

Given a training set $S = \{(x_i, y_i)\}, x \in \mathbb{R}^n, y \in \{-1, 1\}$

1. Initialize $w^0 = 0 \in \mathbb{R}^n$

2. For epoch = 1 ... T:
   1. Pick a random example $(x_i, y_i)$ from the training set $S$

3. Return final $w$
Stochastic gradient descent for SVM

Given a training set $\mathcal{S} = \{(x_i, y_i)\}$, $x \in \mathbb{R}^n$, $y \in \{-1, 1\}$

1. Initialize $w^0 = 0 \in \mathbb{R}^n$

2. For epoch $= 1 \ldots T$:
   
   1. Pick a random example $(x_i, y_i)$ from the training set $\mathcal{S}$

   2. Treat $(x_i, y_i)$ as a full dataset and take the derivative of the SVM objective at the current $w^{t-1}$ to be $\nabla J_t(w^{t-1})$

3. Return final $w$

$$J(w) = \frac{1}{2}w^Tw + C \sum_i \max(0, 1 - y_i w^T x_i)$$
Stochastic gradient descent for SVM

Given a training set $S = \{(x_i, y_i)\}, x \in \mathbb{R}^n, y \in \{-1, 1\}$

1. Initialize $w^0 = 0 \in \mathbb{R}^n$
2. For epoch $= 1 \ldots T$:
   1. Pick a random example $(x_i, y_i)$ from the training set $S$
   2. Treat $(x_i, y_i)$ as a full dataset and take the derivative of the SVM objective at the current $w^{t-1}$ to be $\nabla J^t(w^{t-1})$

\[
J^t(w) = \frac{1}{2} w^T w + C \sum \max (0, 1 - y_i w^T x_i)
\]

3. Return final $w$
Stochastic gradient descent for SVM

Given a training set $S = \{(x_i, y_i)\}$, $x \in \mathbb{R}^n$, $y \in \{-1,1\}$

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\[
J^t(w) = \frac{1}{2} w^T w + C \max (0, 1 - y_i w^T x_i)
\]

3. Update: $w^t \leftarrow w^{t-1} - \gamma_t \nabla J^t(w^{t-1})$

3. Return final $w$
Stochastic gradient descent for SVM

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   3. Update: $w^t \leftarrow w^{t-1} - \gamma_t \nabla J^t(w^{t-1})$

3. Return final $w$

This algorithm is guaranteed to converge to the minimum of $J$ if $\gamma_t$ is small enough. Why? The objective $J(w)$ is a $\textbf{convex}$ function
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Gradient Descent vs SGD
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Many more updates than gradient descent, but each individual update is less computationally expensive.

Stochastic Gradient descent
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Given a training set \( S = \{(x_i, y_i)\} \), \( x \in \mathbb{R}^n \), \( y \in \{-1, 1\} \)

1. Initialize \( w^0 = 0 \in \mathbb{R}^n \)

2. For epoch = 1 ... T:
   1. Pick a random example \((x_i, y_i)\) from the training set \( S \)
   2. Treat \((x_i, y_i)\) as a full dataset and take the derivative of the SVM objective at the current \( w^{t-1} \) to be \( \nabla J^t(w^{t-1}) \)
   3. Update: \( w^t \leftarrow w^{t-1} - \gamma_t \nabla J^t(w^{t-1}) \)

3. Return final \( w \)

What is the derivative of the hinge loss with respect to \( w \)?
(The hinge loss is not a differentiable function!)
Hinge loss is **not** differentiable!

What is the derivative of the hinge loss with respect to $w$?

$$J^t(w) = \frac{1}{2} w^T w + C \max(0, 1 - y_i w^T x_i)$$
Detour: Sub-gradients

Generalization of gradients to non-differentiable functions
(Remember that every tangent is a hyperplane that lies below the function for convex functions)

Informally, a sub-tangent at a point is any hyperplane that lies below the function at the point.
A sub-gradient is the slope of that line
Sub-gradients

Formally, a vector $g$ is a subgradient to $f$ at point $x$ if

$$f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y$$
Sub-Gradients

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$f$ is differentiable at $x_1$

Tangent at this point

$$f(x_1) + g_1^T(x - x_1)$$

$g_1$ is a gradient at $x_1$
Sub-gradients

Formally, a vector $g$ is a subgradient to $f$ at point $x$ if

$$f(y) \geq f(x) + g^T(y - x) \quad \text{for all } y$$

$f$ is differentiable at $x_1$
Tangent at this point

$$f(x_1) + g_1^T(x - x_1)$$

$g_1$ is a gradient at $x_1$

$g_2$ and $g_3$ is are both subgradients at $x_2$

[Example from Boyd]
Sub-gradient of the SVM objective

\[ J^t(w) = \frac{1}{2} w^T w + C \max (0, 1 - y_i w^T x_i) \]

**General strategy:** First solve the max and compute the gradient for each case.
Sub-gradient of the SVM objective

\[ J^t(w) = \frac{1}{2} w^T w + C \max (0, 1 - y_i w^T x_i) \]

**General strategy:** First solve the max and compute the gradient for each case

\[ \nabla J^t = \begin{cases} 
  w & \text{if } \max (0, 1 - y_i w^T x_i) = 0 \\
  w - Cy_i x_i & \text{otherwise}
\end{cases} \]
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Stochastic sub-gradient descent for SVM

\[ \nabla J_t = \begin{cases} 
  w & \text{if } \max(0, 1 - y_i w^T x_i) = 0 \\
  w - C y_i x_i & \text{otherwise}
\end{cases} \]

Given a training set \( S = \{(x_i, y_i)\}, \ x \in \mathbb{R}^n, \ y \in \{-1,1\} \)

1. Initialize \( w = 0 \in \mathbb{R}^n \)

3. Return \( w \)
Stochastic sub-gradient descent for SVM

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Given a training set \( S = \{(x_i, y_i)\}, \ x \in \mathbb{R}^n, y \in \{-1, 1\} \)

1. Initialize \( w = 0 \in \mathbb{R}^n \)
2. For epoch = 1 … T:
   1. For each training example \((x_i, y_i)\in S:\)

      Update \( w \leftarrow w - \gamma_t \nabla J^t \)

3. Return \( w \)
Stochastic sub-gradient descent for SVM

\[ \nabla J^t = \begin{cases} 
  w & \text{if } \max(0, 1 - y_i w^T x_i) = 0 \\
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Given a training set \( S = \{(x_i, y_i)\}, \mathbf{x} \in \mathbb{R}^n, y \in \{-1, 1\} \)

1. Initialize \( w = 0 \in \mathbb{R}^n \)

2. For epoch = 1 ... T:
   1. For each training example \((x_i, y_i)\) \( \in S \):
      - If \( y_i w^T x_i \leq 1 \),
        \[ w \leftarrow (1 - \gamma_i) w + \gamma_i C y_i x_i \]
      - else
        \[ w \leftarrow (1 - \gamma_i) w \]

3. Return \( w \)
Stochastic sub-gradient descent for SVM

Given a training set $S = \{(\mathbf{x}_i, y_i)\}$, $\mathbf{x} \in \mathbb{R}^n$, $y \in \{-1, 1\}$

1. Initialize $\mathbf{w} = 0 \in \mathbb{R}^n$

2. For epoch $= 1 \ldots T$:
   1. For each training example $(\mathbf{x}_i, y_i) \in S$:
      
      If $y_i \mathbf{w}^T \mathbf{x}_i \leq 1$,
      
      $\mathbf{w} \leftarrow (1- \gamma_t) \mathbf{w} + \gamma_t C y_i \mathbf{x}_i$

      else
      
      $\mathbf{w} \leftarrow (1- \gamma_t) \mathbf{w}$

3. Return $\mathbf{w}$
Stochastic sub-gradient descent for SVM

Given a training set \( S = \{(x_i, y_i)\}, x \in \mathbb{R}^n, y \in \{-1, 1\} \)

1. Initialize \( w = 0 \in \mathbb{R}^n \)

2. For epoch = 1 … T:
   1. Shuffle the training set
   2. For each training example \((x_i, y_i) \in S\):
      - If \( y_i w^T x_i \leq 1 \),
        \[
        w \leftarrow (1 - \gamma_i) w + \gamma_i C y_i x_i
        \]
      - else
        \[
        w \leftarrow (1 - \gamma_i) w
        \]

3. Return \( w \)

\( \gamma_i \): learning rate, many tweaks possible
Convergence and learning rates

With enough iterations, it will converge in expectation

Provided the step sizes are “square summable, but not summable”

- Step sizes $\gamma_t$ are positive
- Sum of squares of step sizes over $t = 1$ to $\infty$ is not infinite
- Sum of step sizes over $t = 1$ to $\infty$ is infinity

Some examples: $\gamma_t = \frac{\gamma_0}{1 + \frac{\gamma_0 t}{C}}$ or $\gamma_t = \frac{\gamma_0}{1 + t}$
Convergence and learning rates

- Number of iterations to get to accuracy within $\epsilon$

- For strongly convex functions, $N$ examples, $d$ dimensional:
  - Gradient descent: $O(Nd \ln(1/\epsilon))$
  - Stochastic gradient descent: $O(d/\epsilon)$

- More subtleties involved, but SGD is generally preferable when the data size is huge
Convergence and learning rates

• Number of iterations to get to accuracy within $\epsilon$

• For strongly convex functions, N examples, d dimensional:
  – Gradient descent: $O(Nd \ln(1/\epsilon))$
  – Stochastic gradient descent: $O(d/\epsilon)$

• More subtleties involved, but SGD is generally preferable when the data size is huge

• Recently, many variants that are based on this general strategy, targeting multilayer neural networks
  – Examples: Adagrad, momentum, Nesterov’s accelerated gradient, Adam, RMSProp, etc...
Stochastic sub-gradient descent for SVM

Given a training set \( S = \{(x_i, y_i)\}, x \in \mathbb{R}^n, y \in \{-1, 1\} \)

1. Initialize \( w = 0 \in \mathbb{R}^n \)

2. For epoch = 1 ... T:
   1. Shuffle the training set
   2. For each training example \((x_i, y_i) \in S:\)
      * If \( y_i w^T x_i \leq 1, \)
      \[ w \leftarrow (1-\gamma_t) \ w + \gamma_t \ C \ y_i \ x_i \]
      * else
      \[ w \leftarrow (1-\gamma_t) \ w \]

3. Return \( w \)
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Given a training set $S = \{(x_i, y_i)\}, \mathbf{x} \in \mathbb{R}^n, y \in \{-1,1\}$

1. Initialize $\mathbf{w} = 0 \in \mathbb{R}^n$

2. For epoch = 1 ... T:
   1. Shuffle the training set
   2. For each training example $(x_i, y_i) \in S$:
       If $y_i \mathbf{w}^T \mathbf{x}_i \leq 1$, 
       $$\mathbf{w} \leftarrow (1 - \gamma_t) \mathbf{w} + \gamma_t C y_i \mathbf{x}_i$$
       else
       $$\mathbf{w} \leftarrow (1 - \gamma_t) \mathbf{w}$$

3. Return $\mathbf{w}$

Compare with the Perceptron update:
If $y \mathbf{w}^T \mathbf{x} \leq 0$, update $\mathbf{w} \leftarrow \mathbf{w} + r y \mathbf{x}$
Perceptron vs. SVM

• Perceptron: Stochastic sub-gradient descent for a different loss
  – No regularization though

\[ L_{\text{Perceptron}}(y, x, w) = \max(0, -yw^T x) \]

• SVM optimizes the hinge loss
  – With regularization

\[ L_{\text{Hinge}}(y, x, w) = \max(0, 1 - yw^T x) \]
SVM summary from optimization perspective

- Minimize regularized hinge loss

- Solve using stochastic gradient descent
  - Very fast, run time does not depend on number of examples
  - Compare with Perceptron algorithm: Perceptron does not maximize margin width
    - Perceptron variants can force a margin
  - Convergence criterion is an issue; can be too aggressive in the beginning and get to a reasonably good solution fast; but convergence is slow for very accurate weight vector

- Other successful optimization algorithms exist
  - Eg: Dual coordinate descent, implemented in liblinear

Questions?